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Spectral Theory of Time Dispersive and Dissipative Systems

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We study linear time dispersive and dissipative systems. Very often such systems are not conservative and the standard spectral theory can not be applied. We develop a mathematically consistent framework allowing (i) to constructively determine if a given time dispersive system can be extended to a conservative one; (ii) to construct that very conservative system—which we show is essentially unique. We illustrate the method by applying it to the spectral analysis of time dispersive dielectrics and the damped oscillator with retarded friction. In particular, we obtain a conservative extension of the Maxwell equations which is equivalent to the original Maxwell equations for a dispersive and lossy dielectric medium.

KEY WORDS: Time dispersive dissipative systems; time dispersive dissipative dielectric medium; Lamb model; conservative extension; spectral theory; Maxwell equations.

1. INTRODUCTION

In this paper we describe a mathematical framework for a spectral theory of linear time dispersive and dissipative (lossy) media, e.g., dielectric media. Here is a concise formulation of the setup. We consider a linear system (medium) whose state is described by a time-dependent *generalized velocity* $v(t)$ taking values in a Hilbert space H_0 with scalar product $(\cdot,\cdot)$. The evolution of v is governed by a linear equation incorporating *retarded friction*

$$m\partial_t v(t) = -iAv(t) - \int_0^\infty a(\tau)\, v(t-\tau)\, d\tau + f(t), \tag{1.1}$$

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where $m>0$ is a positive mass operator in H_0, A is a self-adjoint operator in H_0, $f(t)$ is a time dependent external *generalized force*, and $a(t)$, $t\geqslant 0$, is an operator valued function which we call the *operator valued friction retardation function* [ref. 8, Section 1.6], or just *friction function*. The names "generalized velocity" and "generalized force" are justified when we interpret the real part of the scalar product $\mathrm{Re}\{(v(t), f(t))\}$ as the work done by $f(t)$ per unit of time at instant t. Hence, the total work $\mathcal{W}$ done by $f(t)$ is

$$\mathcal{W}=\int_{-\infty}^{\infty}\mathrm{Re}\{(v(t), f(t))\}\,dt. \tag{1.2}$$

By the same token, the first two terms on the right hand side of (1.1), namely

$$-iAv(t)-\int_0^{\infty}a(\tau)\,v(t-\tau)\,d\tau \tag{1.3}$$

are interpreted as the force which v exerts on itself at time t. This self forcing depends on v through two terms: (i) the instantaneous term $-iAv(t)$ and (ii) the "time dispersive" integral term $\int_0^{\infty}a(\tau)\,v(t-\tau)\,d\tau$ involving values $v(t')$ for $t'<t$. The integral term is interpreted as a (retarded) friction force whose special form reflects two fundamental requirements: (i) time homogeneity and (ii) causality.

If we rescale the variables according to the formulas

$$\tilde{v}=\sqrt{m}v,\quad \Omega=\frac{1}{\sqrt{m}}A\frac{1}{\sqrt{m}},\quad \tilde{a}=\frac{1}{\sqrt{m}}a\frac{1}{\sqrt{m}},\quad \tilde{f}=\frac{1}{\sqrt{m}}f, \tag{1.4}$$

then Equation (1.1) reduces to the special form with m the identity operator, i.e.,

$$\partial_t\tilde{v}(t)=-i\Omega\tilde{v}(t)-\int_0^{\infty}\tilde{a}(\tau)\,\tilde{v}(t-\tau)\,d\tau+\tilde{f}(t). \tag{1.5}$$

We refer to $\Omega=\frac{1}{\sqrt{m}}A\frac{1}{\sqrt{m}}$ as the frequency operator in H_0 since its spectrum gives the resonant frequencies of the system.

We call the system conservative, or non-dispersive, if the friction function vanishes, i.e.,

$$m\partial_t v(t)=-iAv(t)+f(t). \tag{1.6}$$

In this case, it is natural to define the internal energy of the system at time t to be $\frac{1}{2}(v(t), mv(t))$, since it follows from (1.1) that

$$\frac{d}{dt}\frac{1}{2}(v(t), mv(t)) = \mathrm{Re}\{(v(t), f(t))\}, \tag{1.7}$$

where $\mathrm{Re}\{(v(t), f(t))\}$ is the instantaneous rate of work (by assumption). In the non-conservative (dispersive) case, we continue to interpret $\frac{1}{2}(v(t), mv(t))$ as the internal energy but note that (1.7) is not true in general. Instead,

$$\frac{d}{dt}\frac{1}{2}(v(t), mv(t)) = \mathrm{Re}\{(v(t), f(t))\} - \mathrm{Re}\left\{\left(v(t), \int_0^\infty a(\tau)\, v(t-\tau)\, d\tau\right)\right\}. \tag{1.8}$$

We interpret the second term of this expression as the instantaneous rate of "work done by the system on itself," or more properly the negative rate of energy dissipation due to friction.

In all physical models of which we are aware the time dispersive term a arises as a phenomenological description of a linear coupling between the system and some other "hidden" degrees of freedom. Our belief in the conservation of energy suggests an equivalent description with the hidden system described by a vector w in a (different) Hilbert space H_1, such that the extension

$$V(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \tag{1.9}$$

evolves according to a conservative equation like (1.6).

The main result of this paper is that under a physically reasonable hypothesis it is possible to construct a conservative extension of the original system (1.1) in the following form

$$m\partial_t v(t) = -iAv(t) - \mathrm{i}\Gamma w(t) + f(t), \quad m > 0, \ A \text{ is self-adjoint}, \tag{1.10}$$
$$m_1\partial_t w(t) = -i\Gamma^\dagger v(t) - iA_1 w(t), \quad m_1 > 0, \ A_1 \text{ is self-adjoint}, \tag{1.11}$$

where $w \in H_1$, the Hilbert space of "hidden" variables, m_1 and A_1 describe respectively the mass operator and the generator of internal dynamics on H_1, and $\Gamma : H_1 \to H_0$ is a coupling operator between the "hidden" and "observable" variables. While it may be conceptually clear that such extensions *should* exist for reasonable linear models, it is not immediately obvious how to determine from the given Equation (1.1) if such an extension *does* exist. Nonetheless, we show there is a natural intrinsic condition—non-positivity of the total work done by the friction force—which is both necessary and sufficient for such an extension. Furthermore, we shall see that the Hilbert space H_1 of "hidden variables" as well as the operators $\Omega_1 = \frac{1}{\sqrt{m_1}} A_1 \frac{1}{\sqrt{m_1}}$ and Γ in (1.10)–(1.11) are essentially uniquely determined by the friction function.

It would be interesting and natural to study (1.10), (1.11), and thus (1.1), with a random force $f(t)$ as a fluctuation-dissipation model similar to the Langevin equation.[8] However, the importance of the subject and the efforts needed to conduct such a study are worthy of a separate publication.

For a homogeneous dielectric medium described by a scalar-valued frequency dependent electric susceptibility $\hat{\chi}(\omega)$, and magnetic permeability $\mu = 1$, the conservative extension takes the form (in common notations)

$$\begin{aligned} \partial_t \mathbf{H}(\mathbf{r}, t) &= -\nabla \times \mathbf{E}(\mathbf{r}, t), \\ \partial_t \mathbf{E}(\mathbf{r}, t) &= \nabla \times \mathbf{H}(\mathbf{r}, t) - \int_{-\infty}^{\infty} \sqrt{m_{\hat{\chi}} n_{\hat{\chi}}(\sigma)} \Psi(\mathbf{r}, t, \sigma)\, d\sigma - 4\pi \mathbf{J}(\mathbf{r}, t), \\ m_{\hat{\chi}} \partial_t \Psi(\mathbf{r}, t, \sigma) &= -\mathrm{i} m_{\hat{\chi}} \sigma \Psi(\mathbf{r}, t, \sigma) + \sqrt{m_{\hat{\chi}} n_{\hat{\chi}}(\sigma)} \mathbf{E}(\mathbf{r}, t), \end{aligned} \tag{1.12}$$

where the field $\Psi(\mathbf{r}, t, \sigma)$ describes "hidden" variables, which one may view as a "string of dipoles" with string coordinate $\sigma \in \mathbb{R}$ attached at every space point $\mathbf{r}$. The parameters $n_{\hat{\chi}}(\sigma)$ and $m_{\hat{\chi}}$ are related to the electric susceptibility $\hat{\chi}(\omega)$ as follows:

$$\begin{aligned} n_{\hat{\chi}}(\sigma) = 4 \mathrm{Im}\left\{\sigma \hat{\chi}(\sigma)\right\} \geqslant 0, \quad m_{\hat{\chi}}^{-1} = \int_{-\infty}^{\infty} n_{\hat{\chi}}(\sigma)\, d\sigma, \\ 4\pi \omega \hat{\chi}(\omega) = \lim_{\eta \to +0} \int_{-\infty}^{\infty} \frac{n_{\hat{\chi}}(\sigma)}{\sigma - \omega - i\eta}\, d\sigma, \qquad \mathrm{Im}\ \zeta > 0. \end{aligned} \tag{1.13}$$

In the second equation of (1.12), the term $\int_{-\infty}^{\infty} \sqrt{m_{\hat{\chi}} n_{\hat{\chi}}(\sigma)} \Psi(\mathbf{r}, t, \sigma)\, d\sigma$ is $4\pi \times$ the displacement current, and thus the electric polarization is related to Ψ by

$$\mathbf{P}(\mathbf{r}, t) = \frac{1}{4\pi} \int_{-\infty}^{t} \int_{-\infty}^{\infty} \sqrt{m_{\hat{\chi}} n_{\hat{\chi}}(\sigma)} \Psi(\mathbf{r}, \tau, \sigma),\, d\sigma d\tau \tag{1.14}$$

assuming $\mathbf{P} \equiv 0$ at $t = -\infty$. In addition, the fields $\mathbf{H}(\mathbf{r}, t)$, $\mathbf{D}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t)$ are required to be divergence free, i.e.

$$\begin{aligned} &\nabla \cdot \mathbf{H}(\mathbf{r}, t) = 0, \\ &\nabla \cdot \mathbf{E}(\mathbf{r}, t) + \int_{-\infty}^{t} \int_{-\infty}^{\infty} \sqrt{m_{\hat{\chi}} n_{\hat{\chi}}(\sigma)} \nabla \cdot \Psi(\mathbf{r}, \tau, \sigma)\, d\sigma\, d\tau = 0. \end{aligned} \tag{1.15}$$

A system of extended Maxwell equations similar to (1.12) were proposed in ref. 18.

In many physical models there is an instantaneous contribution to the friction. Thus we assume throughout that $a(t)$, $t \geqslant 0$ is of the form

$$a(t) = \alpha_\infty \delta(t) + \alpha(t), \qquad \text{where } \alpha(t) \text{ is continuous for } t \geqslant 0, \tag{1.16}$$

with α_∞ self-adjoint (any anti-self-adjoint piece can be incorporated in $-iA$). The representation (1.16) explicitly distinguishes the instantaneous component $\alpha_\infty \delta(t)$ of the friction function from the retarded component described by $\alpha(t)$. It is possible to extend some of the results below to $\alpha(t)$ which are operator valued measures or distributions, but to simplify the exposition such extensions—which are physically somewhat esoteric anyway—will not be considered. On the other hand, the instantaneous component $\alpha_\infty \delta(t)$ is completely natural and occurs in many examples.

To state the central condition of this paper, let us consider the total work $\mathcal{W}_{\text{fr}}$ done by the friction force, which may be written as follows:

$$\mathcal{W}_{\text{fr}} = -\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (v(t), a_e(t-\tau) v(\tau))\, dt\, d\tau, \tag{1.17}$$

where

$$a_e(t) = 2\alpha_\infty \delta(t) + \begin{cases} \alpha(t) & \text{if } t > 0, \\ \operatorname{Re}\{\alpha(+0)\} & \text{if } t = 0, \\ \alpha^\dagger(-t) & \text{if } t < 0, \end{cases} \quad -\infty < t < \infty. \tag{1.18}$$

Notice that $a_e(-t) = a_e^\dagger(t)$. The *precise condition which distinguishes systems with conservative extensions is that $\mathcal{W}_{\text{fr}}$ should be non-positive*, that is,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (v(t), a_e(t-\tau) v(\tau))\, dt\, d\tau \geqslant 0, \tag{1.19}$$

for every function $v(t)$, signifying the ultimate conversion of mechanical energy into heat—the transport of energy from relevant (observable) degrees of freedom to hidden ones. We refer to (1.19) as the *power dissipation condition*.

Scalar functions $a_e(t)$ which satisfy (1.19), known as *positive definite* functions, are familiar from Bochner's theorem on the Fourier transform of a finite positive measure. In fact, one construction of a conservative extension to (1.1) is based on Theorem 3.2, an operator valued generalization of Bochner's theorem.

It is often useful to work in the frequency domain, and the power dissipation condition (1.19) may be formulated there as well. To do so we formally apply the Fourier transform defined by

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \hat{v}(\omega) \, d\omega, \quad \hat{v}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} v(t) \, dt \tag{1.20}$$

to recast the time evolution equation (1.1) as

$$\omega m \hat{v}(\omega) = \left[A - i\hat{a}(\omega)\right] \hat{v}(\omega) + i\hat{f}(\omega). \tag{1.21}$$

Here

$$\hat{a}(\omega) = \alpha_\infty + \hat{\alpha}(\omega), \quad \hat{\alpha}(\omega) = \int_0^{\infty} \alpha(\tau) e^{i\omega\tau} \, d\tau \tag{1.22}$$

is a formal object, and may not be defined pointwise since we do not assume integrability of $\alpha(\tau)$. However, if it *is* defined pointwise, we see—in view of Bochner's theorem—that the power dissipation condition (1.19) becomes

$$\hat{a}_e(\omega) \geqslant 0 \quad \text{for all } \omega, \tag{1.23}$$

where

$$\hat{a}_e(\omega) = 2\text{Re}\left\{\hat{a}(\omega)\right\} = \hat{a}(\omega) + \hat{a}^\dagger(\omega). \tag{1.24}$$

That is, $\hat{a}_e(\omega)$ is a positive semi-definite operator,

$$\left(v, \hat{a}_e(\omega) v\right) \geqslant 0 \quad \text{for all } v \in \mathcal{H}. \tag{1.25}$$

We refer to (1.23) also as the power dissipation condition. If the Fourier transform is not defined pointwise the situation is a bit more delicate, but in effect $\hat{a}_e(\omega)\,d\omega$ is a non-negative operator valued measure (see Theorem 3.13.)

In many examples $a(t)$ is Hermitian for every t. In those cases the real and imaginary parts of $\hat{a}(\omega)$,

$$\operatorname{Re}\{\hat{a}(\omega)\} = \frac{1}{2}\left[\hat{a}(\omega) + \hat{a}^{\dagger}(\omega)\right], \qquad \operatorname{Im}\{\hat{a}(\omega)\} = \frac{1}{2i}\left[\hat{a}(\omega) - \hat{a}^{\dagger}(\omega)\right] \tag{1.26}$$

are respectively even and odd functions of ω given by the cos and sin transforms of a,

$$\begin{aligned} \operatorname{Re}\{\hat{a}(\omega)\} &= \alpha_{\infty} + \int_0^{\infty} \alpha(\tau)\cos(\omega\tau)\, d\tau, \\ \operatorname{Im}\{\hat{a}(\omega)\} &= \int_0^{\infty} \alpha(\tau)\sin(\omega\tau)\, d\tau. \end{aligned} \tag{1.27}$$

In the important special case of a conservative system (1.6), Equation (1.21) reduces to

$$\omega m \hat{v}(\omega) = A\hat{v}(\omega) + i\hat{f}(\omega). \tag{1.28}$$

When the external force vanishes ($f = 0$), a time harmonic solution $v(t) = V_{\omega}\, e^{-i\omega t}$ with $\hat{v}(\omega') = V_{\omega}\delta(\omega - \omega')$ is obtained for V_{ω} which solves the spectral problem

$$\omega m V_{\omega} = A V_{\omega}. \tag{1.29}$$

This eigenvalue problem is part of the standard spectral theory of self-adjoint operators, and its analysis is instrumental to the study of the non-dispersive evolution (1.6). In particular, we remind the reader that the eigenvectors V_{ω} – which may lie in a proper extension of the Hilbert space – are m-orthogonal for different ω and form a basis of the Hilbert space $\mathcal{H}$.

In contrast, a time harmonic solution $v(t) = v_{\omega} e^{-i\omega t}$ to (1.1) satisfies

$$\omega m v_{\omega} = \hat{A}(\omega)\, v_{\omega}, \quad \text{with } \hat{A}(\omega) = A - i\hat{a}(\omega). \tag{1.30}$$

In non-trivial examples, the operator $\hat{A}(\omega)$ is typically non-self-adjoint. Consequently the spectral theory for $\hat{A}(\omega)$ may be rather complicated, even in the simplest examples, with a finite dimensional Hilbert space, m the identity matrix, and $\hat{A}(\omega)$ a finite square matrix. For instance, $\hat{A}(\omega)$

may not be diagonalizable as it may have nontrivial blocks in Jordan form. Consequently, the genuine eigenvectors of $\hat{A}(\omega)$ may not form a basis. In many problems of interest, including continuum dielectric media, the Hilbert space is infinite dimensional and the operator $\hat{A}(\omega)$ is unbounded, in addition to being non-self-adjoint. Thus, in general it seems to be very difficult to analyze the eigenvalue problem (1.30). There are some results on the completeness of so-called root vectors for dissipative linear operators, (ref. 5, Chapter V), but the conditions of those statements—for instance, compactness of the imaginary part of the operator—are too restrictive and are often not satisfied in problems of interest.

Compounding these difficulties, even if the complicated spectral analysis of non-self-adjoint $\hat{A}(\omega)$ were somehow addressed for fixed ω, another problem would arise since $\hat{A}(\omega)$ depends on the spectral parameter ω. Thus there is no obvious relation between solutions to (1.30) for different values of ω. In particular, the vectors v_ω need not be orthogonal and it is not clear that they span the Hilbert space.

However, for physically meaningful examples $\hat{A}(\omega)$ is not an arbitrary ω dependent non self-adjoint operator, but rather one with certain properties which allow some kind of spectral analysis. The relevant properties, as we will see below, are analyticity and a dissipation condition for complex ω in the upper half-plane.

The focus of this article is the construction of a spectral theory for a wide class of dispersive and dissipative systems – described by (1.1), (1.21), or (2.37) below – under the assumption of a suitable power dissipation condition. As indicated above, our approach is based on the observation that dispersion and dissipation are caused in most physical models by coupling with degrees of freedom which are "hidden" from observation. Consequently, we consider a spectral theory for a dispersive/dissipative system to be a realization of the system as a proper projection of a conservative extension to which the standard spectral theory may be applied. As we show, *the power dissipation condition is necessary and sufficient for such a conservative extension and determines the (minimal) conservative extension uniquely.*

The paper is organized as follows. In Section 2 we recall the analysis of a general conservative system of the form (1.10), (1.11), which we take as an abstract model for a system with "hidden variables." In particular we show that positive power dissipation holds for any system with a conservative extension. We also describe the *admittance operator* reformulation of (1.1) and (1.10), (1.11). In Section 3, we discuss the main mathematical results which demonstrate that positive power dissipation is *equivalent* to the existence of a conservative extension. There are two approaches to constructing a conservative extension, proceeding by either the time (1.1) or the frequency

(1.21) representations and based on the classical Bochner's Theorem 3.1 and Herglotz–Nevanlinna Theorems 3.8, 3.9 respectively. In Section 4, we summarize in concise form the schemes by which one may construct a conservative extention of a given dispersive/dissipative system including (i) the space of "hidden" variables; (ii) a self-adjoint operator describing the internal dynamics of the "hidden" variables; (iii) an operator coupling the "hidden" variables with the "observable" variables. In Section 5, we give examples of conservative extensions for general scalar dispersive systems and homogeneous isotropic dielectrics, obtaining an extended conservative system which is equivalent to Maxwell's equations for a dispersive and lossy dielectric medium. In forthcoming work we shall describe in greater detail the application of the techniques presented here to dispersion in dielectric media. In Section 6, we discuss dissipation, or loss of energy, a phenomenon which can arise in systems with dispersion. We present sufficient conditions on $a(t)$ for solutions $v(t)$ to (1.1) to exhibit dissipation, that is $\lim_{t\to\infty} \|v(t)\| = 0$. Finally, Section 7 is devoted to proofs of results in Section 3 and related constructions.

2. MODELLING HIDDEN DEGREES OF FREEDOM

In this section we introduce and analyze abstract models for hidden degrees of freedom, showing in particular that the power dissipation condition holds for the truncation of any conservative system. In addition, we recall the equivalent description of a linear system in terms of its admittance operator and discuss the admittance for truncations.

2.1. Abstract Model: Preliminary Analysis

Consider a conservative system with degrees of freedom (variables) divided into two classes: the *observable variables*, denoted v, and the *hidden variables*, denoted w. For instance, v might describe the polarization of a dielectric medium at different space points, while components of w account for microscopic degrees of freedom which give rise to the material relations. We assume that v and w take values respectively in Hilbert spaces H_0 and H_1, and that the combination

$$V = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathcal{H} = H_0 \oplus H_1, \quad v \in H_0, \quad w \in H_1 \tag{2.1}$$

describes a *conservative* system governed by the following evolution equation:

$$\mathcal{M}\partial_t V(t) = -i\mathcal{A}V(t) + F(t), \tag{2.2}$$

where $\mathcal{M} > 0$ and $\mathcal{A}$ are self-adjoint operators in $\mathcal{H}$. Furthermore we assume—and this is key—that the external force $F(t)$ is of the form

$$F(t) = \begin{bmatrix} f(t) \\ 0 \end{bmatrix} =: T^\dagger f(t) \tag{2.3}$$

indicating that we may excite directly only the degrees of freedom corresponding to the observable variables v.

In (2.3) we have introduced the operator:

$$T = \begin{bmatrix} I_{H_0} & 0 \end{bmatrix} : \mathcal{H} \to H_0, \quad TT^\dagger = I_{H_0}, \quad T^\dagger T = P_{H_0}, \tag{2.4}$$

where I_{H_0} and P_{H_0} are respectively the identity operator in H_0 and the orthogonal projection onto $\operatorname{ran} T^\dagger = H_0 \oplus \{0\}$ in $\mathcal{H}$. Notice that $T^\dagger : H_0 \to \mathcal{H}$ is an isometric injection from H_0 into $\mathcal{H}$, and we can recast Equations (2.2) and (2.3) as

$$\mathcal{M} \partial_t V(t) = -i\mathcal{A} V(t) + T^\dagger f(t), \quad V(t) \in \mathcal{H}, \quad f(t) \in H_0. \tag{2.5}$$

The operator T is an example of an *isometric truncation*:

Definition 2.1. Given two Hilbert spaces $\mathcal{H}$ and H_0, a bounded linear operator $T : \mathcal{H} \to H_0$ is called an *isometric truncation* of $\mathcal{H}$ to H_0 if $TT^\dagger = I_{H_0}$.

Observe that when $H_0 \subseteq \mathcal{H}$ the orthogonal projection P_{H_0} of $\mathcal{H}$ onto H_0 is an isometric truncation. *The conservative system as described by Equation (2.5) with $\mathcal{M} > 0$ $\mathcal{A}$ self-adjoint, and T an isometric truncation is of the most general form we consider in this paper.*

We are specifically interested in conservative extensions to (1.1) of the form (1.10–1.11), corresponding to the special case of Equation (2.5) with operators $\mathcal{M}$ and $\mathcal{A}$ in the following block-matrix form:

$$\begin{aligned} \mathcal{M} &= \begin{bmatrix} m & 0 \\ 0 & I_{H_1} \end{bmatrix}, \quad m > 0, \\ \mathcal{A} &= \begin{bmatrix} A & \Gamma \\ \Gamma^\dagger & \Omega_1 \end{bmatrix}, \quad A^\dagger = A, \quad \Omega_1^\dagger = \Omega_1, \quad \Gamma : H_1 \to H_0, \end{aligned} \tag{2.6}$$

where I_{H_1} is the identity operator in H_1. In physical models with unbounded operators, the block decomposition (2.6) may be somewhat formal, since there remains the question of specifying the domain of each operator.

In general we consider A and Ω_1 which are self-adjoint with domains $\mathcal{D}(A)$ and $\mathcal{D}(\Omega_1)$ respectively. For the time being, we take the operator Γ—which evidently provides the coupling between the observable and hidden degrees of freedom—to be a *bounded* map from H_1 to H_0. In this case, the natural domain for $\mathcal{A}$ is the direct sum $\mathcal{D}(A)\oplus\mathcal{D}(\Omega_1)$, and with this choice the operator is self-adjoint. We consider unbounded Γ below to describe systems with non-vanishing instantaneous friction, e.g., a damped oscillator. In that case, Γ will be a map from $\mathcal{D}(\Omega_1)$ to H_0, and we shall have to specify the domain of $\mathcal{A}$ carefully.

We can represent a formal solution to (2.2)—using the rescaling (1.4)—by the formula:

$$\begin{aligned} V(t) &= \int_0^{\infty} \mathcal{M}^{-\frac{1}{2}} e^{-i\mathcal{A}_m\tau} \mathcal{M}^{-\frac{1}{2}} F(t-\tau)\, d\tau \\ &= \int_{-\infty}^{t} \mathcal{M}^{-\frac{1}{2}} e^{-i\mathcal{A}_m(t-\tau)} \mathcal{M}^{-\frac{1}{2}} F(\tau)\, d\tau, \quad \mathcal{A}_m = \mathcal{M}^{-\frac{1}{2}} \mathcal{A} \mathcal{M}^{-\frac{1}{2}}, \end{aligned} \tag{2.7}$$

assuming the system was *at rest* with $V(t)=0$, $F(t)=0$ in distant past. In particular, we note that

$$V(t) = \mathcal{M}^{-\frac{1}{2}} e^{-i\mathcal{A}_m t} \mathcal{M}^{-\frac{1}{2}} F_0, \quad t>0, \tag{2.8}$$

for a pulse force $F(t)=F_0\delta(t)$, and

$$V(t) = \mathcal{M}^{-\frac{1}{2}} e^{-i\mathcal{A}_m t} \left[\int_{-\infty}^{\infty} e^{-i\mathcal{A}_m\tau} \mathcal{M}^{-\frac{1}{2}} F(\tau)\, d\tau + o(1)\right] \tag{2.9}$$

for $t\to\infty$ if, say, $\int_{-\infty}^{\infty} \left\| \mathcal{M}^{-\frac{1}{2}} F(\tau) \right\| < \infty$.

To obtain an effective equation for the evolution of v, let us look at the block form of the evolution equation (2.2) with $\mathcal{M}$, $\mathcal{A}$ and F respectively satisfying (2.6) and (2.3),

$$m\partial_t v(t) = -iAv(t) - i\Gamma w(t) + f(t), \tag{2.10}$$

$$\partial_t w(t) = -i\Gamma^{\dagger} v(t) - i\Omega_1 w(t). \tag{2.11}$$

Note that $-i\Gamma^{\dagger}v(t)$ plays the role of an external force in Equation (2.11). Thus we can solve for w as in Equation (2.7), obtaining

$$w(t) = -i\int_0^{\infty} e^{-i\Omega_1\tau}\Gamma^{\dagger} v(t-\tau)\, d\tau. \tag{2.12}$$

Plugging this into Equation (2.10) yields

$$m\partial_t v(t) = -iAv(t) - \int_0^\infty \Gamma e^{-i\Omega_1\tau}\Gamma^\dagger v(t-\tau)\,d\tau + f(t). \tag{2.13}$$

The dispersive evolution equation (2.13) is of the form (1.1) with friction function

$$a(t) = \Gamma e^{-i\Omega_1 t}\Gamma^\dagger, \quad \text{for } t > 0, \tag{2.14}$$

and describes the evolution of the observable variable v. We note in particular that the instantaneous friction α_∞ vanishes—since $a(0) = \Gamma\Gamma^\dagger$ is finite—and the extended friction function $a_e(t)$ is

$$a_e(t) = \Gamma e^{-i\Omega_1 t}\Gamma^\dagger \quad \text{for } -\infty < t < \infty. \tag{2.15}$$

Notice that for $a_e(t)$ of this form we have

$$\int_{-\infty}^\infty \int_{-\infty}^\infty (v(t), a_e(t-\tau)v(\tau))\,dtd\tau = \left\| \int_{-\infty}^\infty e^{i\Omega_1 t}\Gamma^\dagger v(t)\,dt \right\|^2 \tag{2.16}$$

readily implying that the extended friction function $a_e(t) = \Gamma e^{i\Omega_1\tau}\Gamma^\dagger$ satisfies the power dissipation condition (1.19). *The central point of this work is that Ω_1 and Γ needed to satisfy (2.14) can be re-constructed from the friction function $a(t)$.*

2.2. Unbounded Coupling and Instantaneous Friction

Let us briefly consider how to define the abstract model (2.6) with unbounded Γ, a necessary step in the description of systems with non-vanishing instantaneous friction ($\alpha_\infty \neq 0$ in (1.16)). The following discussion is by nature a bit technical; we direct the reader to the case of the damped oscillator in Section 5 for an explicit example which may provide clarification.

Suppose Γ is given as a map $\Gamma : \mathcal{D}(\Omega_1) \to H_0$ which is Ω_1-bounded, i.e.,

$$\|\Gamma\phi\|^2 \leqslant C \left\| \left(\Omega_1 + iI_{H_1}\right)\phi \right\|^2, \tag{2.17}$$

where I_{H_1} is the identity operator and $C < \infty$. We continue to take A and Ω_1 to be self adjoint on their respective domains $\mathcal{D}(A)$ and $\mathcal{D}(\Omega_1)$ as above. The operator $\Gamma\Phi_R$, with

$$\Phi_R = \frac{R}{\sqrt{\Omega_1^2 + R^2 I_{H_1}}} \qquad \text{for } R > 0 \tag{2.18}$$

is a bounded map from H_1 to H_0, with $\|\Gamma\Phi_R\| \lesssim R$ as $R \to \infty$. We denote the adjoint of this map by $\Phi_R\Gamma^\dagger$, although $\Gamma^\dagger$ has not been defined,[3] and define $\mathcal{A}$ as the limit

$$\mathcal{A}\begin{bmatrix} v \\ w \end{bmatrix} := \lim_{R\to\infty} \begin{bmatrix} Av + \Gamma\Phi_R w \\ \Phi_R\Gamma^\dagger v + \Omega_1\Phi_R w \end{bmatrix}, \tag{2.19}$$

on the domain $\mathcal{D}(\mathcal{A})$ of vectors $[v, w]^T$ such that the limit exits.

As things stand, it is not clear if the resulting operator is self-adjoint, or even that $\mathcal{D}(\mathcal{A})$ is dense. To proceed we require an additional assumption – (2.24) below – which guarantees self-adjointness. To state that condition consider the map

$$S := \Gamma\left(\Omega_1 + iI_{H_1}\right)^{-1}, \tag{2.20}$$

which is bounded and satisfies

$$\left(\Omega_1 - iI_{H_1}\right)\Phi_R S^\dagger v = \Phi_R\Gamma^\dagger v \quad \text{for any } v \in H_0. \tag{2.21}$$

Therefore the limit

$$\lim_{R\to\infty} \Phi_R\Gamma^\dagger v + \Omega_1\Phi_R w = \lim_{R\to\infty} \left(\Omega_1 - iI_{H_1}\right)\Phi_R\left(\frac{\Omega_1}{\Omega_1 - iI_{H_1}} w + S^\dagger v\right). \tag{2.22}$$

exists if and only if

$$\frac{\Omega_1}{\Omega_1 - iI_{H_1}} w + S^\dagger v \in \mathcal{D}(\Omega_1), \tag{2.23}$$

[3] It may be defined as a map from H_0 to a proper extension of H_1, namely the space $\mathcal{D}(B)^*$ of conjugate linear functionals on $\mathcal{D}(B)$, but we do not use this fact here.

which is equivalent to saying that $w = \phi - S^{\dagger} v$ with $\phi \in \mathcal{D}(\Omega_1)$. Thus $[v, w]^T \in \mathcal{D}(\mathcal{A})$ if and only if $w = \phi - S^{\dagger} v$ with $\phi \in \mathcal{D}(\Omega_1)$ and

$$Gv := \lim_{R \to \infty} \Gamma \Phi_R S^{\dagger} v = \lim_{R \to \infty} \Gamma \frac{1}{\Omega_1 - i I_{H_1}} \Phi_R \Gamma^{\dagger} v \tag{2.24}$$

exists. We *require* of Γ and Ω_1 that the limit (2.24) exists for every $v \in \mathcal{D}(A)$, and defines an A bounded operator G with A bound less than one – i.e., there are $\delta < 1$ and $\beta_\delta > 0$ such that

$$\|Gv\| \leqslant \delta \left\| \left(A + i\beta_\delta I_{H_0}\right) v \right\|. \tag{2.25}$$

Under these assumptions

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= \left\{ \begin{bmatrix} v \\ w \end{bmatrix} : v \in \mathcal{D}(A) \text{ and } \lim_{R \to \infty} \Phi_R \Gamma^{\dagger} v + \Omega_1 \Phi_R w \text{ exists} \right\} \\ &= \left\{ \begin{bmatrix} v \\ \phi - Sv \end{bmatrix} : v \in \mathcal{D}(A) \text{ and } \phi \in \mathcal{D}(\Omega_1) \right\}, \end{aligned} \tag{2.26}$$

and the operator $\mathcal{A}$ is self-adjoint:

Proposition 2.2. Let A, Ω_1 be self adjoint operators on the Hilbert spaces H_0, H_1 with domains $\mathcal{D}(A)$, $\mathcal{D}(\Omega_1)$ respectively. Suppose that $\Gamma : \mathcal{D}(\Omega_1) \to H_0$ is such that (2.17) holds and the limit (2.24) exists for all $v \in \mathcal{D}(A)$, defining an operator G for which the bound (2.25) holds. If $\mathcal{A}$ is defined by (2.19) on the domain $\mathcal{D}(\mathcal{A}) \subset H_0 \oplus H_1$ specified in (2.26), then $\mathcal{A}$ is self-adjoint.

The proof of this proposition is elementary. It is obvious that $\mathcal{A}$ is symmetric, so to prove self-adjointness we need only to show that $\mathcal{A}^{\dagger} V = \pm i V$ implies $V = 0$, which is an easy exercise.

For the operator $\mathcal{A}$ defined in this way, the evolution equations (2.10)–(2.11) imply that

$$\partial_t \Phi_R w(t) = -i \Phi_R \Gamma^{\dagger} v(t) - i \Omega_1 \Phi_R w(t) \tag{2.27}$$

and thus

$$\Phi_R w(t) = -i \int_0^{\infty} e^{-i \Omega_1 \tau} \Phi_R \Gamma^{\dagger} v(t - \tau)\, d\tau. \tag{2.28}$$

Therefore

$$m\partial_t v(t) = -iAv(t) - \lim_{R\to\infty}\int_0^\infty \Gamma e^{-i\Omega_1\tau}\Phi_R^2\Gamma^\dagger v(t-\tau)\,d\tau + f(t). \quad (2.29)$$

While this equation is formally similar to (2.13), we note that the resulting friction function

$$a_e(t) = \lim_{R\to\infty}\Gamma\frac{e^{-i\Omega_1 t}}{\frac{\Omega_1^2}{R^2}+I_{H_1}}\Gamma^\dagger, \qquad \text{for } -\infty < t < \infty \quad (2.30)$$

is defined only as a distribution,

$$\int_{-\infty}^\infty a_e(t)v(t)\,\mathrm{d}t = \lim_{R\to\infty}\int_{-\infty}^\infty \Gamma\frac{e^{-i\Omega_1 t}}{\frac{\Omega_1^2}{R^2}+I_{H_1}}\Gamma^\dagger v(t)\,dt, \text{ for } v\in C_c(\mathbb{R},H_0), \quad (2.31)$$

and may not in fact be a function. In particular, there may be non-vanishing instantaneous friction. However $a_e(t)$ is a relatively tame distribution; it may be expressed as a second order differential operator applied to a (strongly) continuous function

$$a_e(t) = \left(-\frac{d^2}{dt^2}+1\right)Se^{-i\Omega_1 t}S^\dagger \text{ with } S = \Gamma(\Omega_1 + iI_{H_1})^{-1}. \quad (2.32)$$

In a key example, $H_1 = L^2(\mathbb{R}, H_0)$, the space of square integrable H_0-valued functions on the real line, and Ω_1 is multiplication by the independent variable, $\Omega_1\psi(x) = x\psi(x)$. Given any positive operator α_∞, say bounded, on H_0 we define

$$[\Gamma\psi](x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty \sqrt{\alpha_\infty}\psi(x)\,dx, \quad (2.33)$$

which is Ω_1 bounded since

$$[S\psi](x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty \frac{1}{x+i}\sqrt{\alpha_\infty}\psi(x)\,dx \quad (2.34)$$

is bounded. Thus

$$Se^{-i\Omega_1 t}S^{\dagger} = \frac{1}{\pi}\int_{-\infty}^{\infty} dx \frac{e^{-ixt}}{x^2+1}\alpha_{\infty} = e^{-|t|}\alpha_{\infty} \tag{2.35}$$

and

$$a_e(t) = \left(-\frac{d^2}{dt^2}+1\right) Se^{-i\Omega_1 t}S^{\dagger} = 2\alpha_{\infty}\delta(t), \tag{2.36}$$

purely instantaneous friction. We shall return to this example in our discussion of the damped oscillator below.

2.3. Linear Response and the Admittance Operator

In the linear response theory, e.g. ref. 8, Section 3, a system is often characterized by its *admittance operator* $\mathfrak{A}(\omega): H_0 \to H_0$, defined by the relation

$$\hat{v}(\omega) = \mathfrak{A}(\omega)\hat{f}(\omega), \quad \hat{v}(\omega), \ \hat{f}(\omega) \in H_0 \tag{2.37}$$

at real frequencies ω. Here we discuss the admittance formulation of dissipative dispersive systems satisfying the power dissipation condition (1.19) and truncated conservative systems described by (2.5). For this purpose, it is useful to recast the linear response equation (2.37) in the domain of complex frequencies $\zeta = \omega + i\eta$, $\eta > 0$, which corresponds to replacing the Fourier transform by a Fourier–Laplace transform.

We assume that the system governed by (1.1) is at rest for all negative times, i.e.,

$$v(t) = 0, \quad f(t) = 0, \ t \leqslant 0, \tag{2.38}$$

and define the Laplace transforms

$$\hat{v}(\zeta) \ = \int_0^{\infty} e^{i\zeta t}v(t)dt, \quad \hat{f}(\zeta) = \int_0^{\infty} e^{i\zeta t} f(t)\,dt, \tag{2.39}$$

$$\hat{a}(\zeta) = \int_{[0,\infty)} e^{i\zeta t}a(t)\,dt = \alpha_{\infty} + \int_{[0,\infty)} e^{i\zeta t}\alpha(t)\,dt, \tag{2.40}$$

on the domain {Im $\zeta > 0$}. It is convenient to assume that $\alpha(t)$ is bounded as $t \to \infty$ so that $\hat{a}(\zeta)$ is well defined (see Condition 3.5 below). Under the Laplace transform, the evolution equation (1.1) is transformed into the following identity

$$\zeta m\hat{v}(\zeta)=\left[A-i\hat{a}(\zeta)\right]\hat{v}(\zeta)+i\hat{f}(\zeta),\quad \zeta=\omega+i\eta,\ \ \eta=\mathrm{Im}\ \zeta>0. \tag{2.41}$$

We have Re $\hat{a}(\zeta)\geqslant 0$ (for Im $\zeta>0$), since

$$\mathrm{Re}\ \left(v,\hat{a}(\zeta)v\right)=\frac{\mathrm{Im}\ \zeta}{2}\int_{[0,\infty)}\int_{[0,\infty)}(v,a_e(t-\tau)v)\,\mathrm{e}^{i\zeta t}\mathrm{e}^{-i\bar{\zeta}\tau}dt\,d\tau\geqslant 0 \tag{2.42}$$

for any $v\in H_0$ by the power dissipation condition (1.19). Hence, the operator $\zeta m-A+i\hat{a}(\zeta)$ is invertible, and

$$\hat{v}(\zeta)=\mathfrak{A}_{m,A,a}(\zeta)\hat{f}(\zeta),\quad \zeta=\omega+i\eta,\ \ \eta=\mathrm{Im}\ \zeta>0, \tag{2.43}$$

$$\mathfrak{A}_{m,A,a}(\zeta)=i\left[\zeta m-A+i\hat{a}(\zeta)\right]^{-1}. \tag{2.44}$$

The equation (2.43) generalizes (2.37) to a certain extent since it is an identity for analytic functions in the upper half plane. Note that

$$\begin{aligned}\mathrm{Re}\left\{\mathfrak{A}_{m,A,a}(\zeta)\right\}=&\ \frac{1}{2}\left[\mathfrak{A}_{m,A,a}(\zeta)+\mathfrak{A}^{\dagger}_{m,A,a}(\zeta)\right]\\=&\ \mathfrak{A}_{m,A,a}(\zeta)\left\{\mathrm{Im}\ \zeta m+\mathrm{Re}\,\hat{a}(\zeta)\right\}\mathfrak{A}^{\dagger}_{m,A,a}(\zeta)\geqslant 0.\end{aligned} \tag{2.45}$$

which expresses the power dissipation condition (1.19) in terms of the admittance operator $\mathfrak{A}_{m,A,a}$.

The *admittance equation* (2.43) provides an essentially equivalent description of the system (1.1). In particular, the various operators in (1.1) can be readily recovered from $\mathfrak{A}_{m,A,a}(\omega)$ by the relations

$$\begin{aligned}m^{-1}=-\lim_{\eta\to\infty}\eta\mathfrak{A}_{m,A,a}(i\eta),\quad &A=-\lim_{\eta\to\infty}\sqrt{m}\,\mathrm{Im}\ \mathfrak{A}^{-1}_{m,A,a}(i\eta)\sqrt{m},\\ &\hat{a}(\zeta)=i(\zeta m-A)+\left[\mathfrak{A}_{m,A,a}(\zeta)\right]^{-1}.\end{aligned} \tag{2.46}$$

Hence, the admittance operator $\mathfrak{A}_{m,A,a}(\omega)$ carries all the information about the system initially described by the triplet $\left\{m,A,\hat{a}(\zeta)\right\}$. Very often the admittance equation (2.37) is a preferred form, since the admittance $\mathfrak{A}(\omega)$ may be measured experimentally more readily than m, A or $\hat{a}(\omega)$.

There are also several technical advantages to the admittance formulation (2.43). First, the quantities $\hat{v}(\zeta)$, $\hat{a}(\zeta)$, $\mathfrak{A}(\zeta)$, $\hat{f}(\zeta)$ are analytic functions in the upper half-plane Im $\zeta > 0$, whereas their time counterparts may be more singular functions. In addition (2.43) has the advantage that auxiliary operators which appear in the analysis are *bounded*. In particular, we shall see that for $\alpha_\infty \neq 0$ the admittance formulation permits us to avoid the subtleties required for an unbounded coupling Γ.

Consider now a system which is the truncation of a general conservative system of the form (2.5). Under the Laplace transform (2.5) is transformed into

$$-i\zeta \mathcal{M}\hat{V}(\zeta) = -i\mathcal{A}\hat{V}(\zeta) + T^\dagger \hat{f}(\zeta), \quad \text{Im } \zeta > 0 \tag{2.47}$$

which is easily solved for $\hat{V}$,

$$\hat{V}(\zeta) = i(\zeta\mathcal{M} - \mathcal{A})^{-1} T^\dagger \hat{f}(\zeta), \ \text{Im } \zeta > 0. \tag{2.48}$$

Multiplying of the both sides of (2.48) by the isometric truncation T: $\mathcal{H} \to H_0$ yields

$$\hat{v}(\zeta) = \mathfrak{A}(\zeta)\hat{f}(\zeta), \quad \zeta = \omega + i\eta, \ \eta = \text{Im } \zeta > 0, \tag{2.49}$$

$$\mathfrak{A}(\zeta) = iT(\zeta\mathcal{M} - \mathcal{A})^{-1} T^\dagger. \tag{2.50}$$

Notice also that the admittance operator as defined in (2.49) satisfies

$$\begin{aligned} \text{Re}\{\mathfrak{A}(\zeta)\} &= \frac{\mathfrak{A}(\zeta) + \mathfrak{A}^\dagger(\zeta)}{2} \\ &= (\text{Im } \zeta)\, T(\zeta\mathcal{M} - \mathcal{A})^{-1} \mathcal{M} \left[T(\zeta\mathcal{M} - \mathcal{A})^{-1}\right]^\dagger. \end{aligned} \tag{2.51}$$

This identity together with $\mathcal{M} > 0$ implies

$$\text{Re}\{\mathfrak{A}(\zeta)\} \geqslant 0. \tag{2.52}$$

For operators $\mathcal{M}, \mathcal{A}$ in the block-matrix form (2.6) we have

$$\mathfrak{A}(\zeta) = i\left(\zeta m - A - \Gamma\left(\zeta I_{H_1} - \Omega_1\right)^{-1} \Gamma^\dagger\right)^{-1} \tag{2.53}$$

as may be easily verified. Comparing (2.53) and (2.44) gives the following formula for the Laplace transform $\hat{a}(\zeta)$ of the friction function for a truncated conservative system

$$\hat{a}(\zeta) = i\Gamma\left(\zeta I_{H_1} - \Omega_1\right)^{-1}\Gamma^{\dagger}, \tag{2.54}$$

which could also be verified by directly transforming (2.14).

A second approach to the central construction of this work is based upon the observation that the Hilbert space $\mathcal{H}$ and the operator triple $\mathcal{M}$, $\mathcal{A}$, T in Equation (2.50) can be reconstructed from the admittance $\mathfrak{A}(\zeta)$. Alternatively H_1, Γ, and Ω_1 appearing in (2.54) can be reconstructed from $\hat{a}(\zeta)$.

3. BASES FOR A CONSERVATIVE EXTENSION

We now describe how, given a dispersive system in the form (1.1) or (1.21), one can reconstruct the hidden degrees of freedom. Of course, the resulting mathematical reconstruction is initially devoid of physical interpretation. However, the reader should bear in mind that one usually knows a given evolution equation of the form (1.1) involves "hidden" degrees of freedom with a natural physical interpretation. Generally, the abstract extension may be interpreted therefore in a physically concrete way.

For example in a classical dielectric medium, the time dispersion comes from the material relation between the electric displacement $\mathbf{D}(\mathbf{r}, t)$ at a point $\mathbf{r}$ and the electric field $\mathbf{E}(\mathbf{r}, t)$, namely

$$\mathbf{D}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) + 4\pi\mathbf{P}(\mathbf{r}, t), \qquad \text{where } \hat{\mathbf{P}}(\mathbf{r}, \omega) = \hat{\chi}(\omega)\,\hat{\mathbf{E}}(\mathbf{r}, \omega) \tag{3.1}$$

with $\hat{\mathbf{P}}$ and $\hat{\mathbf{E}}$ the time Fourier transforms of $\mathbf{P}$ and $\mathbf{E}$ and $\hat{\chi}(\omega)$ the frequency dependent electric susceptibility tensor. The relations (3.1) can be viewed as a macroscopic picture of the interactions between the electromagnetic field and the material medium, and we can naturally interpret the hidden variables that arise in the spectral theory of this equation as a model for the material medium. For the time being, however, we ignore such considerations (which we take up in Section 5 wherein we discuss specific models). Herein we focus on the abstract, mathematical, structure of the hidden variables.

Thus given a dissipative system, defined by either the evolution equation (1.1) or its frequency counterpart (1.21), our main problem is to find a larger conservative system, governed respectively by the equation (1.6) or its frequency counterpart (1.28), that reduces correspondingly to (1.1) or (1.21) upon integrating out the hidden variables. Clearly the power dissipation condition in the form (1.19) for the friction function $a(t)$ or in the form (2.52) for the admittance function $\mathfrak{A}(\omega)$ is a necessary condition for the existence of such an extension, as indicated by the relations (2.15)–(2.16) and (2.50)–(2.51) obtained for the truncation of a conservative system. Remarkably, we shall see it is also a sufficient condition.

The equality (2.15) is a possible base for the construction of such a conservative extension. Thus, we pose the following problem: given a friction function $a(t)$ satisfying the dissipation condition (1.19) find an operator Γ and a self-adjoint operator Ω_1 for which the equality (2.15) holds. Then the desired conservative system is (2.1)–(2.3). Alternatively, we may start with the relation (2.50) and ask, given the admittance $\mathfrak{A}(\zeta)$ with $\mathrm{Re}\mathfrak{A}(\zeta) \geqslant 0$, whether we can find an isometric truncation $T\colon \mathcal{H} \to H$ along with self-adjoint $\mathcal{M} > 0$ and $\mathcal{A}$ for which (2.50) holds. These two closely related approaches each lead to constructions of an extension, based respectively on operator versions of the following fundamental results: Bochner's Theorem, (ref. 1, Section 60) (ref. 19, Section XI, ref. 13, Theorem 2), and the Herglotz–Nevanlinna Theorems (ref. 1, Section 69; ref. 7, ref. 10, Section 32.1, Theorem 2 and 3). In the second approach, the Naimark Theorem on positive operator valued measures plays a key role (ref. 1, Vol. II, Appendix I, Section I; ref. 13, Appendix, Section 2, Theorem I).

3.1. Approach via Bochner's Theorem

We begin by recalling (ref. 1, Section 60; ref. 19, Section XI; ref. 13, Theorem 2) and (ref. 14, Theorem IX.9):

Theorem 3.1 (Bochner). A complex-valued continuous function $s(t)$ of $-\infty < t < \infty$, is representable as

$$s(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-it\sigma}\, dN(\sigma) \tag{3.2}$$

with a non-decreasing, right-continuous bounded function $N(\sigma)$ if and only if $s(t)$ is positive-definite in the following sense

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(t-\tau)\,\overline{\varphi(t)}\varphi(\tau)\, dt\, d\tau \geqslant 0 \tag{3.3}$$

for every continuous function $\varphi(t)$ with compact support.

The measure $dN(\sigma)$ may be realized as the spectral measure associated to the vector $\chi_{\mathbb{R}}$ and the operator Ω_1 of multiplication by σ on $L^2(\mathbb{R}, dN)$. Thus

$$s(t) = \Gamma e^{-it\Omega_1} \Gamma^\dagger \tag{3.4}$$

with Γ the following rank one operator

$$\Gamma \psi = \int \psi(\sigma)\, dN(\sigma)\colon L^2(\mathbb{R}, dN) \to \mathbb{C}. \tag{3.5}$$

Hence, Bochner's Theorem may be seen as a special case of the following result, which we prove below in Section 6.

Theorem 3.2. Let $\mathcal{B}(H_0)$ be the space of all bounded linear operators in H_0. Then a strongly continuous $\mathcal{B}(H_0)$-valued function $a_e(t)$, $-\infty < t < \infty$, is representable as

$$a_e(t) = \Gamma e^{-it\Omega_1} \Gamma^\dagger, \tag{3.6}$$

with Ω_1 a self-adjoint operator on a Hilbert space H_1 and $\Gamma\colon H_0 \to H_1$ a bounded linear map, if and only if $a(t)$ satisfies the dissipation condition (1.19) for every continuous H_0 valued function $v(t)$ with compact support. If the space H_1 is minimal—in the sense that the linear span

$$\left\langle g(\Omega_1)\, \Gamma^\dagger v : g \in C_c(\mathbb{R}),\, v \in H_0 \right\rangle \tag{3.7}$$

is dense in H_1—then the triplet $\{H_1, \Omega_1, \Gamma\}$ is determined uniquely up to an isomorphism.

Remark 3.3. In fact, it is sufficient to assume that $a_e(t)$ is locally bounded and strongly measurable, strong continuity then follows from (3.6).

Remark 3.4. $\left\langle g(\Omega_1)\, \Gamma^\dagger v\colon g \in C_c(\mathbb{R}),\, v \in H_0 \right\rangle$ denotes the linear span, i.e., the subspace of linear combinations of finitely many elements of the form $g(\Omega_1)\, \Gamma^\dagger v$.

Our proof of this theorem is a very elementary generalization of the proof of Bochner's Theorem given in ref. 14. Nonetheless we are not aware that Theorem 3.2 has appeared previously in the literature.

Theorem 3.2 provides a basis for constructing a conservative extension of a dispersive system without instantaneous friction: given a system described by a vector v in a Hilbert space H_0 and governed by a dissipative evolution (1.6) with a strongly continuous friction function, we simply represent it as the restriction of a conservative system in the block-matrix form (2.6) with Ω_1 and Γ the operators obtained from Theorem 3.2.

The construction afforded by Theorem 3.2 is sufficiently general to describe most systems of interest, excepting those with instantaneous friction. For such a system we must admit a friction "function" $a(t)$ which is not strongly continuous. For most cases of interest though it is sufficient to assume the following condition.

Condition 3.5. **(friction function)**. The friction function $a(t)$ is of the form

$$a(t) = \alpha_\infty \delta(t) + \alpha(t), \tag{3.8}$$

where α_∞ is a bounded non-negative operator in H_0 and $\alpha(t)$ a strongly continuous and bounded $\mathcal{B}(H_0)$-valued function for $t \geqslant 0$, i.e.

$$0 \leqslant \alpha_\infty \leqslant C I_{H_0}, \quad C < \infty; \ \sup_{t \geqslant 0} \|\alpha(t)\| < \infty. \tag{3.9}$$

The extension $a_e(t) = 2\alpha_\infty \delta(t) + \alpha_e(t)$, $-\infty < t < \infty$ of the function $a(t)$ is defined by the formula (1.18).

The following result for a friction function satisfying Condition 3.5 will be useful. We give its proof below in Section 6.

Theorem 3.6. Suppose that the friction function $a(t)$ satisfies Condition 3.5. Then its extension $a_e(t) = 2\alpha_\infty \delta(t) + \alpha_e(t)$, $-\infty < t < \infty$, is representable as

$$a_e(t) = \operatorname*{Dlim}_{R \to \infty} \Gamma e^{-it\Omega_1} \left(\Gamma \Phi_R^2 \right)^\dagger, \quad \Phi_R^2 = \left(\frac{\Omega_1^2}{R^2} + I_{H_1} \right)^{-1} \tag{3.10}$$

with Ω_1 a self-adjoint operator on H_1 and $\Gamma : \mathcal{D}(\Omega_1) \to H_0$ an Ω_1-bounded linear map, if and only if $a(t)$ satisfies the dissipation condition (1.19) for every continuous H_0 valued function $v(t)$ with compact support. If the space H_1 is minimal—in the sense that

$$\left\langle (\Gamma g(\Omega_1))^\dagger v : g \in C_c(\mathbb{R}), v \in H_0 \right\rangle \tag{3.11}$$

is dense in H_1—then the triplet $\{H_1, \Omega_1, \Gamma\}$ is determined uniquely up to an isomorphism.

Remark 3.7. Here Dlim indicates the distributional limit, i.e.

$$\int_{-\infty}^{\infty} a_e(t)\, v(t)\, dt = \lim_{R\to\infty} \int_{-\infty}^{\infty} \Gamma e^{-it\Omega_1} \Phi_R^2 \Gamma^\dagger v(t)\, dt \tag{3.12}$$

for every smooth H_0 valued function $v(t)$ with compact support. In essence we have $a_e(t) = \Gamma e^{-it\Omega_1}\Gamma^\dagger$, but this expression is ambiguous so we introduce a sort of principle value by regularizing with Φ_R^2.

3.2. Approach via Herglotz–Nevanlinna Theorems

The approach via Bochner's Theorem just outlined is quite straightforward and adequate for many purposes, however there is an equally useful method which works in the frequency domain and makes use of analytic function theory. This second approach is based on an alternative description of the friction function $a(t)$ through its Laplace transform $\hat{a}(\zeta)$ as defined above in Equation (2.40).

Condition 3.5 for the friction function readily implies that

$$\hat{a}(\zeta) = \alpha_\infty + \hat{\alpha}(\zeta), \quad \left\|\hat{\alpha}(\zeta)\right\| \leqslant \frac{\sup_{t\geqslant 0} \|\alpha(t)\|}{\mathrm{Im}\zeta}, \quad \mathrm{Im}\ \zeta > 0. \tag{3.13}$$

One advantage of $\hat{a}(\zeta)$ over $a(t)$ is that it is an analytic function, even if $a(t)$ has the singular term $\alpha_\infty \delta(t)$. Formally the Laplace Transform $\hat{a}(\zeta)$ becomes the Fourier Transform $\hat{a}(\omega)$ for $\zeta = \omega$ with real ω.

The power dissipation condition (1.19) implies that $\mathrm{Im}\ \hat{a}(\zeta) \geqslant 0$ for $\mathrm{Im}\ \zeta > 0$, as we have seen in (2.42). Therefore for each $v \in H$, $\zeta \mapsto (v, i\hat{a}(\zeta)\, v)$ is an analytic map of the upper half plane into itself. There is a classical representation theory for such maps, (ref. 1, Section 59; ref. 7, ref. 10, Section 32.3 Theorems 2 and 3), which provides a tool for the construction of a conservative extension governed by (2.10) and (2.11).

Theorem 3.8. **(Nevanlinna)**. Every analytic function $g(\zeta)$ in the upper half-plane $\mathrm{Im}\ \zeta > 0$ whose imaginary part is everywhere non-negative and which satisfies the growth condition

$$\limsup_{\eta\to+\infty} \eta\, |g(i\eta)| < \infty \tag{3.14}$$

can be expressed uniquely in the form

$$g(\zeta) = \int_{-\infty}^{\infty} \frac{dN(\sigma)}{\sigma - \zeta}, \tag{3.15}$$

where $N(\sigma)$ is a non-decreasing, right-continuous, bounded function such that

$$\int_{-\infty}^{\infty} dN(\sigma) = \limsup_{\eta \to +\infty} \eta \mathrm{Im}\{g(i\eta)\} < \infty. \tag{3.16}$$

In fact, Theorem 3.8 is a special case of the following result.

Theorem 3.9. Every analytic function $g(\zeta)$ in the half-plane $\{\mathrm{Im}\zeta > 0\}$ whose imaginary part is everywhere non-negative can be expressed uniquely in the form

$$g(\zeta) = \xi + \rho\zeta + \int_{-\infty}^{\infty} \frac{1+\sigma\zeta}{\sigma - \zeta} d\tilde{N}(\sigma), \tag{3.17}$$

where $\tilde{N}(\sigma)$ is a non-decreasing, right-continuous, bounded function, ξ is real, and $\rho \geqslant 0$.

Remark 3.10. The measures from Theorems 3.9 and 3.8 are related by the identity $dN(\sigma) = \left(1+\sigma^2\right) d\tilde{N}(\sigma)$, and the former result is obtained from the later by noting that (3.14) implies that $\rho = 0$ and that $\left(1+\sigma^2\right) d\tilde{N}(\sigma)$ is a finite measure.

Returning to the analysis of $a(t)$, suppose first that the instantaneous friction vanishes, $\alpha_\infty = 0$. Then in view of Equation (3.13) we have

$$\hat{a}(\zeta) = \hat{\alpha}(\zeta) \quad \text{and} \quad \left\|\hat{a}(\zeta)\right\| \leqslant \frac{\sup_t \|\alpha(t)\|}{\mathrm{Im}\zeta}. \tag{3.18}$$

Hence, given $v \in H_0$, the function $\left(v, i\hat{a}(\zeta) v\right)$ satisfies the hypotheses of Theorem 3.8 and consequently there is a finite Borel measure $dN_{v,v}(\sigma)$ such that

$$\left(v, i\hat{a}(\zeta) v\right) = \int_{-\infty}^{\infty} \frac{dN_{v,v}(\sigma)}{\sigma - \zeta}. \tag{3.19}$$

For each pair $v, w \in H_0$, we define the "off-diagonal" measures $dN_{v,w}(\sigma)$ via polarization,

$$\begin{aligned} dN_{v,w}(\sigma) = \frac{1}{4}\big[& dN_{v+w,v+w}(\sigma) - dN_{v-w,v-w}(\sigma) \\ & - i dN_{v+iw,v+iw}(\sigma) + i dN_{v-iw,v-iw}(\sigma)\big], \end{aligned} \tag{3.20}$$

so that

$$\left(v, i\hat{a}(\zeta) w\right) = \int_{-\infty}^{\infty} \frac{dN_{v,w}(\sigma)}{\sigma - \zeta} \quad \text{for all } v, w \in H_0. \tag{3.21}$$

Using the measures $dN_{v,w}$ we define, for each $\sigma \in \mathbb{R}$, a non-negative quadratic form

$$Q_\sigma(v, w) = \int_{(-\infty,\sigma]} dN_{v,w}\left(\sigma'\right). \tag{3.22}$$

Because

$$Q_\sigma(v, v) \leqslant \int_{-\infty}^{\infty} dN_{v,v}(\sigma) = \lim \sup_{\eta \to \infty} \eta \mathrm{Im}\left\{\left(v, i\hat{a}(i\eta) v\right)\right\} \leqslant \left(\sup_t \|\alpha(t)\|\right) \|v\|^2, \tag{3.23}$$

we see that these forms are bounded. Thus for each $\sigma \in \mathbb{R}$ there is a non-negative bounded operator $K(\sigma)$ which satisfies $Q_\sigma(v, w) = (v, K(\sigma) w)$. It is easy to see that this *generalized spectral family* of operators satisfies the following condition.

Condition 3.11. (generalized spectral family).

1. $K(\sigma)$, $\sigma \in \mathbb{R}$ are bounded non-negative operators in H_0.
2. $K(\sigma) \leqslant K(\lambda)$ for $\sigma < \lambda$; $K(\sigma + 0) = K(\sigma)$; $\mathrm{stlim}_{\sigma \to -\infty} K(\sigma) = 0$; $K(+\infty) = \mathrm{stlim}_{\sigma \to \infty} K(\sigma)$ exists and is bounded.

There is a fundamental result due to Naimark (ref. 11, in Russian; ref. 1, Vol. II, Appendix I, Section I; ref. 13, Appendix, Section 2, Theorem I, includes uniqueness), which provides a canonical representation for generalized spectral families.

Theorem 3.12. (Naimark). Let $K(\sigma)$, $\sigma \in \mathbb{R}$ be a generalized spectral family satisfying Condition 3.11. Then there exist a Hilbert space H_1, a bounded map $\Gamma: H_1 \to H_0$, and a resolution of the identity $E(\sigma)$, $\sigma \in \mathbb{R}$ of H_1 such that

$$K(\sigma) = \Gamma E(\sigma) \Gamma^\dagger, \quad \sigma \in \mathbb{R}. \tag{3.24}$$

If the space H_1 is minimal—in the sense that

$$\left\langle E(\sigma) \Gamma^\dagger v : \sigma \in \mathbb{R}, \ v \in H_0 \right\rangle \tag{3.25}$$

is dense in H_1—then the triplet $\{H_1, \{E(\sigma), \sigma \in \mathbb{R}\}, \Gamma\}$ is determined uniquely up to an isomorphism.

The construction of a conservative extension from these results proceeds as follows. Given $\hat{a}(\zeta)$ with $\alpha_\infty = 0$, we obtain $K(\sigma)$ from Theorem 3.8 and thence Γ and $E(\sigma)$ from Theorem 3.12. Letting Ω_1 be the self-adjoint operator $\int_{-\infty}^{\infty} \sigma dE(\sigma)$, we obtain

$$\begin{aligned} \hat{a}(\zeta) &= -i \int_{-\infty}^{\infty} \frac{1}{\sigma - \zeta} \Gamma dE(\sigma) \Gamma^\dagger \\ &= i\Gamma \left[\zeta I_{H_1} - \Omega_1\right]^{-1} \Gamma^\dagger, \quad \Omega_1 = \int_{-\infty}^{\infty} \sigma dE(\sigma), \end{aligned} \tag{3.26}$$

which is the desired representation for $\hat{a}(\zeta)$ – compare with Equation (2.54). This construction is summarized by the following operator generalization of the Nevanlinna theorem.

Theorem 3.13. Every $\mathcal{B}(H_0)$-valued analytic function $G(\zeta)$ of the upper half plane $\operatorname{Im} \zeta > 0$ with $\operatorname{Im} G(\zeta)$ everywhere a non-negative operator, and which obeys the growth condition

$$\limsup_{\eta \to +\infty} \eta \|G(i\eta)\| < \infty \tag{3.27}$$

can be expressed in the form

$$G(\zeta) = \Gamma \left[\Omega_1 - \zeta I_{H_1}\right]^{-1} \Gamma^\dagger \tag{3.28}$$

with Ω_1 a self-adjoint operator on a Hilbert space H_1 and $\Gamma: H_1 \to H_0$ a bounded map such that

$$\Gamma \Gamma^\dagger v = \lim_{\eta \to +\infty} \eta G(i\eta) v \quad \text{for every} \quad v \in H_0. \tag{3.29}$$

If the space H_1 is minimal—in the sense that

$$\left\langle f(\Omega_1) \Gamma^\dagger v : \ f \in C_c(\mathbb{R}), v \in H_0 \right\rangle \tag{3.30}$$

is dense in H_1—then the triplet $\{H_1, \Omega_1, \Gamma\}$ is determined uniquely up to an isomorphism.

Remark 3.14. Theorem 3.13 and operator generalizations of the full Herglotz–Nevanlinna theorem 3.9 are certainly known to experts, and have seen application, for example, in the theory of self-adjoint extensions of symmetric operators—see, e.g., ref. 4.

Remark 3.15. In the scalar case, the function $g(\zeta)$ in (3.15) may be expressed in terms of the resolvent of the self-adjoint operator $\Omega_1\phi(\sigma) = \sigma\phi(\sigma)$ on $L^2(dN)$,

$$g(\zeta) = \Gamma\left[\Omega_1 - \zeta I_{H_1}\right]^{-1}\Gamma^{\dagger} \tag{3.31}$$

with

$$\Gamma\psi = \int \psi(\sigma)\, dN(\sigma), \quad \Gamma: L^2(\mathbb{R}, dN) \to \mathbb{C} \tag{3.32}$$

the rank one operator as in the discussion of Bochner's Theorem above. Thus Theorem 3.8 may be seen as a special case of Theorem 3.13 (although the latter is in fact a consequence of the former).

For systems with instantaneous friction ($\alpha_\infty \neq 0$), we shall make use of the following result, which is derived from Theorem 3.9 and the Naimark representation (Theorem 3.12) in the same way as Theorem 3.13. We discuss the proof of this result, and related constructions, in Section 7 below.

Theorem 3.16. Let $G(\zeta)$ be a $\mathcal{B}(H_0)$-valued analytic function of the upper half plane $\mathrm{Im}\ \zeta > 0$ with $\mathrm{Im}\ G(\zeta) \geqslant 0$ everywhere. If

$$\lim_{R\to\infty} G(iR)\, v = i g_\infty v \tag{3.33}$$

exists for all $v \in H_0$ with $g_\infty \geqslant 0$ a non-negative operator, then $G(\zeta)$ can be expressed in the form

$$G(\zeta)v = \lim_{R\to\infty} \Gamma\left[\Omega_1 - \zeta I_{H_1}\right]^{-1}\left(\Gamma\frac{R^2}{\Omega_1^2 + R^2 I_{H_1}}\right)^{\dagger} v \quad \text{for all } v \in H_1 \tag{3.34}$$

with Ω_1 a self-adjoint operator on H_1, and $\Gamma: \mathcal{D}(\Omega_1) \to H_0$ a Ω_1-bounded linear map. If the space H_1 is minimal—in the sense that

$$\left\langle (\Gamma f(\Omega_1))^{\dagger} v : \ f \in C_c(\mathbb{R}),\ v \in H_0 \right\rangle \tag{3.35}$$

is dense in H_1—then the triplet $\{H_1, \Omega_1, \Gamma\}$ is determined uniquely up to an isomorphism.

For a friction function $a(t)$ satisfying Condition 3.5 and, consequently, $\hat{a}(\zeta)$ satisfying (3.13) the function $G(\zeta)=i\hat{a}(\zeta)$ satisfies the hypotheses of Theorem 3.16 with $g_\infty=\alpha_\infty$. Hence, we can apply Theorem 3.16 to obtain operators Γ and Ω_1 which provide a basis for a conservative extension via the abstract model of the previous section. The restriction that g_∞ be self-adjoint is natural in this context, since any anti-Hermitian contribution to $\lim_{R\to\infty}\hat{a}(iR)$ can be absorbed in $-iA$. In the present context, the operator G appearing in Proposition 2.2 is $G=\hat{a}(i)$, which is in fact bounded.

3.3. Spectral Representation of the Admittance Operator

The admittance operator $\mathfrak{A}(\zeta)=i\left[\zeta m-A+i\hat{a}(\zeta)\right]^{-1}$ associated to the dispersive system (1.1) is an analytic function on the domain {Im $\zeta>0$}, and the power dissipation condition takes the form $\mathrm{Re}\,\mathfrak{A}(\zeta)\geqslant 0$ when expressed in terms of $\mathfrak{A}$ (see Equation (2.52).) Furthermore, under condition 3.11, we see that

$$\lim_{\eta\to\infty}\eta\mathfrak{A}(i\eta)=m^{-1}. \tag{3.36}$$

Thus $i\mathfrak{A}(\zeta)$ satisfies the conditions of Theorem 3.13 and therefore has the following representation:

$$\mathfrak{A}(\zeta)=i\Gamma(\zeta-\Omega)^{-1}\Gamma^{\dagger} \tag{3.37}$$

with Ω self-adjoint on a Hilbert space $\mathcal{H}$ and $\Gamma:\mathcal{H}\to H_0$ bounded. In this section we develop a basis for a slightly different representation of the admittance operator, namely

$$\mathfrak{A}(\zeta)=iT(\zeta\mathcal{M}-\mathcal{A})^{-1}T^{\dagger}$$
$$\text{with } T:\mathcal{H}\to H_0 \text{ an isometric truncation.} \tag{3.38}$$

In fact, Equation (3.38) is a consequence of Equation (3.37) and the polar decomposition $\Gamma=T\mathcal{M}^{-\frac{1}{2}}$where $T:\mathcal{H}\to H_0$ is an isometric truncation and $0<\delta\leqslant\mathcal{M}$. The relationship between Equations (3.38) and (3.37) is expressed through the identity $\mathcal{A}=\mathcal{M}^{\frac{1}{2}}\Omega\mathcal{M}^{\frac{1}{2}}$. However, there is some flexibility in the choice of $\mathcal{M}$, which we discuss below. Since $\mathcal{M}^{\frac{1}{2}}\Omega\mathcal{M}^{\frac{1}{2}}$ may not be well defined for unbounded $\mathcal{M}$, we shall require throughout that $\mathcal{M}$ be bounded.

Let us first fix some notation. Given two Hilbert spaces $\mathcal{H}$, H_0 and a bounded linear operator $L:\mathcal{H}\to H_0$ we denote

$$\operatorname{Ker} L=\{V\in\mathcal{H}: LV=0\}, \quad \operatorname{Ran} L=\{LV : V\in\mathcal{H}\}. \tag{3.39}$$

We denote the closure of a subset S in a Hilbert space by $\overline{S}$, and the restriction of L to S by $L|_S$. We need the following elementary facts

$$\operatorname{Ker} L=\left[\operatorname{Ran} L^{\dagger}\right]^{\perp}, \quad \overline{\operatorname{Ran} L}=\left[\operatorname{Ker} L^{\dagger}\right]^{\perp}, \tag{3.40}$$

where $[\cdot]^{\perp}$ denotes the orthogonal complement in the relevant Hilbert space. We refer to the orthogonal direct sum decomposition

$$\mathcal{H}=\overline{\operatorname{Ran} L^{\dagger}}\oplus\operatorname{Ker} L, \qquad \text{where } \overline{\operatorname{Ran} L^{\dagger}}=[\operatorname{Ker} L]^{\perp}, \tag{3.41}$$

as the L-decomposition of the Hilbert space $\mathcal{H}$.

Theorem 3.13 readily implies the following statements regarding the decomposition associated to an operator function $G(\zeta)$ of the type considered there.

Corollary 3.17. Let $G(\zeta): H_0\to H_0$ satisfy all the conditions of Theorem 3.13, so $G(\zeta)=\Gamma(\Omega_1-\zeta I_{\mathcal{H}})^{-1}\Gamma^{\dagger}$. Then we have

$$\operatorname{Ker} G(\zeta)=\operatorname{Ker} G^{\dagger}(\zeta)=\operatorname{Ker}\Gamma^{\dagger} \quad \text{for all} \quad \operatorname{Im}\zeta>0. \tag{3.42}$$

Furthermore, the $\Gamma^{\dagger}$-decomposition of the Hilbert space H_0, i.e.

$$H_0=\tilde{H}_0\oplus\tilde{H}_0^{\perp}, \quad \tilde{H}_0=\left[\operatorname{Ker}\Gamma^{\dagger}\right]^{\perp}=\overline{\operatorname{Ran}\Gamma} \tag{3.43}$$

reduces $G(\zeta)$ in the sense that $G(\zeta)$ has the following block form under (3.43):

$$G(\zeta)=G_{\tilde{H}_0}(\zeta)\oplus 0=\begin{bmatrix} G_{\tilde{H}_0}(\zeta) & 0\\ 0 & 0\end{bmatrix},$$
$$\text{where } G_{\tilde{H}_0}(\zeta)=\tilde{\Gamma}\left[\Omega_1-\zeta I_{H_1}\right]^{-1}\tilde{\Gamma}^{\dagger}:\tilde{H}_0\to\tilde{H}_0, \ \ \tilde{\Gamma}=P_{\tilde{H}_0}\Gamma. \tag{3.44}$$

In addition

$$\operatorname{Ker} G_{\tilde{H}_0}(\zeta)=\operatorname{Ker}\tilde{\Gamma}^{\dagger}=\{0\}, \quad \overline{\operatorname{Ran}\tilde{\Gamma}}=\tilde{H}_0, \quad \operatorname{Im}\zeta>0. \tag{3.45}$$

Corollary 3.17 allows us to extract from the operator function $G(\zeta)$ its nontrivial component and motivates the following definition.

Definition 3.18. For any function $G(\zeta): H_0 \to H_0$ satisfying all the conditions of Theorem 3.13 the pair $\left\{\tilde{H}_0, G_{\tilde{H}_0}(\zeta)\right\}$ defined in Corollary 3.17 is called its *reduced representation.*

For the representation equation (3.38) to hold we shall of course require that the function $G(\zeta) = -i\mathfrak{A}(\zeta)$ is in reduced form. However, to obtain a bounded mass operator, we must strengthen this requirement by assuming there is $\varepsilon > 0$ such that

$$\operatorname{Im} G(i\eta) \geqslant \frac{\varepsilon}{\eta}, \quad \eta > 1, \tag{3.46}$$

which implies that $\Gamma\Gamma^\dagger \geqslant \varepsilon$. Under this additional assumption, we have the following theorem, which provides a basis for a spectral representation of the admittance operator.

Theorem 3.19. Let $G(\zeta)$ be a $\mathcal{B}(H_0)$-valued analytic function of the upper half plane $\operatorname{Im} \zeta > 0$ with $\operatorname{Im} G(\zeta) \geqslant 0$ for every ζ and assume that $G(\zeta)$ obeys the growth conditions equation (3.46) and

$$\limsup_{\eta\to+\infty} \eta \, \|G(i\eta)\| < \infty. \tag{3.47}$$

Then there exists a Hilbert space $\mathcal{H}$ such that

$$G(\zeta) = \Gamma\,[\Omega - \zeta I_{\mathcal{H}}]^{-1}\,\Gamma^\dagger, \quad \text{where } \Omega \text{ is self-adjoint in } \mathcal{H},$$
$$\Gamma: \mathcal{H} \to H_0 \text{ is bounded, and } \Gamma\Gamma^\dagger = \lim_{\eta\to+\infty} \eta G(i\eta) \geqslant \varepsilon. \tag{3.48}$$

Furthermore, if T denotes the isometric truncation $T := \left[\Gamma\Gamma^\dagger\right]^{-\frac{1}{2}} \Gamma$, then $T = U P_{H_0'}$ with $U = T|_{H_0'}$ a unitary map from $H_0' = [\operatorname{Ker} \Gamma]^\perp$ to H_0, and the following representation holds:

$$G(\zeta) = T\,[\mathcal{A}_{\mathcal{M}} - \zeta\mathcal{M}]^{-1}\,T^\dagger\ , \tag{3.49}$$

where $\mathcal{A}_{\mathcal{M}} = \mathcal{M}^{\frac{1}{2}}\Omega\mathcal{M}^{\frac{1}{2}}$ is self adjoint in $\mathcal{H}$ and $\mathcal{M}: \mathcal{H} \to \mathcal{H}$ is any operator of the form

$$\mathcal{M} = m_G \oplus m_1, \tag{3.50}$$

with m_1: Ker $\Gamma \to$ Ker Γ bounded and strictly positive ($m_1 \geqslant \delta I_{\mathrm{Ker}\,\Gamma}$, $\delta > 0$) and

$$m_G = U^{-1}\left[\Gamma\Gamma^{\dagger}\right]^{-\frac{1}{2}} U. \tag{3.51}$$

Note that $\mathcal{M}$ is bounded and strictly positive since $m_G \geqslant \left\|\Gamma\Gamma^{\dagger}\right\|^{-\frac{1}{2}} I_{H_0'}$ and $\|m_G\| \leqslant \frac{1}{\varepsilon}$. If the space $\mathcal{H}$ is minimal—in the sense that

$$\left\langle f(\Omega)\,\Gamma^{\dagger}v\colon\ f \in C_c(\mathbb{R}),\, v \in H_0\right\rangle \tag{3.52}$$

is dense in $\mathcal{H}$—then the triplet $\{\mathcal{H}, \mathcal{M}, \mathcal{A}\}$ is determined uniquely up to an isomorphism and the choice of m_1.

As the reader may easily verify, Theorem 3.19 follows from Theorem 3.13 and the *polar decomposition* (see, e.g., ref. 6, Section VI.7), summarized here:

Theorem 3.20. Let $\Gamma\colon \mathcal{H} \to H_0$ be a bounded linear operator with Ker $\Gamma^{\dagger} = \{0\}$ and let

$$\mathcal{H} = H_0' \oplus H_1, \quad H_0' = [\mathrm{Ker}\,\Gamma]^{\perp}, \quad H_1 = \mathrm{Ker}\,\Gamma \tag{3.53}$$

be the Γ-decomposition of $\mathcal{H}$. Then

$$\Gamma V \neq 0 \quad \text{for any non-zero} \quad V \in H_0' \quad \text{and} \quad \overline{\Gamma H_0'} = H_0, \tag{3.54}$$

$$\Gamma^{\dagger}\Gamma = K P_{H_0'} \quad \text{where} \quad K = \Gamma^{\dagger}\Gamma\Big|_{H_0'} > 0, \tag{3.55}$$

and the following "polar decomposition" holds

$$\Gamma = U K^{\frac{1}{2}} P_{H_0'}, \quad \text{where } U\colon H_0' \to H_0 \text{ is unitary}, \tag{3.56}$$

indicating, in particular, that H_0' is an isometric copy of H_0. In addition,

$$\Gamma\Gamma^{\dagger} = U K U^{-1}, \quad K = \Gamma^{\dagger}\Gamma\Big|_{H_0'} = U^{-1}\Gamma\Gamma^{\dagger}U, \tag{3.57}$$

$$\Gamma = \left[\Gamma\Gamma^{\dagger}\right]^{\frac{1}{2}} U P_{H_0'}, \tag{3.58}$$

and Γ is an isometric truncation if and only if $K = I_{H_0'}$, in which case

$$\Gamma = U P_{H_0'} \ . \tag{3.59}$$

Furthermore, let $\mathcal{K}$ be any operator in $\mathcal{H}$ of the form

$$\mathcal{K} = K \oplus K_1 \qquad \text{with } K_1\colon H_1 \to H_1 \text{ non-negative.} \tag{3.60}$$

Then the following "generalized polar decomposition" holds

$$\Gamma = T\mathcal{K}^{\frac{1}{2}} = T\left[K^{\frac{1}{2}} \oplus K_1^{\frac{1}{2}}\right] , \tag{3.61}$$

where $T = U P_{H_0'}\colon \mathcal{H} \to H_0$ is an isometric truncation. Notice that T and K are uniquely determined by Γ, but $\mathcal{K}$ depends on the choice of K_1.

4. FINAL SCHEMES OF THE CONSTRUCTION OF THE CONSERVATIVE EXTENSION.

In this section we summarize in a concise form the main consequences for time dispersive systems of the analysis of preceding sections. There are two intimately related ways to construct a conservative extension of a dispersive system. The first, the "friction function scheme," is based on the operator-valued friction function $a(t)$ or its Laplace transform $\hat{a}(\zeta)$, whereas the second, the "admittance operator scheme," is based on the admittance operator $\mathfrak{A}(\zeta)$.

4.1. Friction Function Scheme

The original time dispersive system is described by an evolution equation

$$m\partial_t v(t) = -iAv(t) - \int_0^\infty a(\tau)\, v(t-\tau)\, d\tau + f(t), \quad v(t) \in H_0 \tag{4.1}$$

with H_0 the space of system states v describing "observable" variables, A a self-adjoint operator in H_0 describing the internal dynamics, $a(\tau)$: $H_0 \to H_0$ an operator-valued friction function accounting for time dispersion and losses, and $f(t) \in H_0$ a time dependent external force. Based on the notion that the friction function $a(\tau)$ is a result of a coupling between the "observable" variables $v \in H_0$ and some "hidden" variables described by a vector w belonging to a Hilbert space H_1, *we seek a conservative (not time dispersive) extension of Equation (4.1) in the form*

$$m\partial_t v(t) = -iAv(t) - i\Gamma w(t) + f(t), \tag{4.2}$$
$$\partial_t w(t) = -i\Gamma^{\dagger} v(t) - i\Omega_1 w(t), \tag{4.3}$$

where Ω_1 is a self-adjoint operator in H_1 describing the internal dynamics of the "hidden" variables and Γ: $H_1 \to H_0$ is a coupling operator between the "hidden" and "observable" variables. The system (4.2)–(4.3), of course, is expected to reduce to the original system (4.1) after the "hidden" variables $w(t)$ are eliminated by solving Equation (4.3) and plugging the result into Equation (4.2).

Now, the question is how to construct the conservative extension of the form (4.2)–(4.3) based on the original dispersive system (4.1)? As we have seen, a necessary and sufficient condition for such an extension to exist is the *power dissipation condition*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v(t), a_e(t-\tau)v(\tau))\, dt d\tau \geqslant 0 \quad \text{for every } v(t). \tag{4.4}$$

A first possibility is to obtain the triplet $\{H_1, \Gamma, \Omega_1\}$ from the function $a(t)$ via the operator version of the Bochner Theorem 3.2, which represents $a(t)$ as

$$a(t) = \Gamma e^{-it\Omega_1}\Gamma^{\dagger}. \tag{4.5}$$

However, in practice most systems are specified in the frequency domain, and it is more convenient to carry out the construction in that setting.

Taking the Fourier–Laplace transforms of (4.1) and (4.2)–(4.3) with respect to t yields

$$m\zeta\hat{v}(\zeta) = \left[A - i\hat{a}(\zeta)\right]\hat{v}(\zeta) + i\hat{f}(\zeta), \quad \zeta = \omega + i\eta, \quad \eta = \operatorname{Im}\zeta > 0, \tag{4.6}$$

and the conservative system, recast in block matrix form,

$$\zeta\mathcal{M}\hat{V}(\zeta) = \mathcal{A}\hat{V}(\zeta) + i\hat{F}(\zeta), \quad \zeta = \omega + i\eta, \ \eta = \operatorname{Im}\zeta > 0, \tag{4.7}$$
$$\hat{V}(\zeta) = \begin{bmatrix} \hat{v}(\zeta) \\ \hat{w}(\zeta) \end{bmatrix}, \quad \hat{F}(\zeta) = \begin{bmatrix} \hat{f}(\zeta) \\ 0 \end{bmatrix},$$
$$\mathcal{A} = \begin{bmatrix} A & \Gamma \\ \Gamma^{\dagger} & \Omega_1 \end{bmatrix}, \ \mathcal{M} = \begin{bmatrix} m & 0 \\ 0 & I_{H_1} \end{bmatrix}, \quad \hat{v}(\zeta) = T\hat{V}(\zeta), \quad T = \left[\, I_{H_0} \ 0 \,\right].$$

Note that the operator T defined in (4.7) is an isometric truncation of the extended space $H_0 \oplus H_1$ onto the space of observable variables H_0. After the Laplace transform, the power dissipation condition (4.4) becomes

$$\mathrm{Re}\ \hat{a}(\zeta) \geqslant 0 \quad \text{for } \mathrm{Im}\ \zeta > 0, \tag{4.8}$$

which is equivalent to (4.4).

The scheme of the construction is as follows. Given an operator-valued friction function $\hat{a}(\zeta)$ which satisfies the power dissipation condition (4.8), we apply the operator version of the Nevanlinna theorem formulated in Theorems 3.13 and 3.16 to construct a triplet $\{H_1, \Gamma, \Omega_1\}$ giving the representation

$$\hat{a}(\zeta) = i\Gamma\left(\zeta I_{H_1} - \Omega_1\right)^{-1}\Gamma^{\dagger}, \quad \Gamma: H_1 \to H_0, \quad \Omega_1: H_1 \to H_1 \text{ is self-adjoint} \tag{4.9}$$

if $\left\|\hat{a}(i\eta)\right\| = O(\eta^{-1})$, or

$$\hat{a}(\zeta) = \operatorname*{stlim}_{R\to\infty} i\Gamma\left(\zeta I_{H_1} - \Omega_1\right)^{-1}\left(\Gamma\frac{R^2}{\Omega_1^2 + R^2 I_{H_1}}\right)^{\dagger} \tag{4.10}$$

if $\mathrm{stlim}_{\eta\to\infty}\hat{a}(i\eta)$ exists.

In more detail, assuming $\left\|\hat{a}(i\eta)\right\| = \mathrm{O}(\eta^{-1})$, we first obtain the operator version of the Nevanlinna representation

$$\hat{a}(\zeta) = i\int_{-\infty}^{\infty}\frac{1}{\zeta - \sigma}dK(\sigma), \tag{4.11}$$

where $dK(\sigma): H_0 \to H_0$ is an non-negative operator-valued measure over $\mathbb{R}$. Notice that $dK(\sigma)$ is not necessarily a resolution of identity, and for different intervals Δ_1 and Δ_2 in $\mathbb{R}$ the operators $K(\Delta_1) \geqslant 0$ and $K(\Delta_2) \geqslant 0$ may not commute. However, having found $dK(\sigma)$ we apply the Naimark theorem 3.12 and get (i) a Hilbert space H_1; (ii) a resolution of identity $dE(\sigma)$; (iii) an operator $\Gamma: H_1 \to H_0$ such that

$$dK(\sigma) = \Gamma\, dE(\sigma)\,\Gamma^{\dagger}. \tag{4.12}$$

To complete the construction of the triplet $\{H_1, \Gamma, \Omega_1\}$, we define

$$\Omega_1 := \int_{-\infty}^{\infty}\sigma\, dE(\sigma). \tag{4.13}$$

It is when applying the Naimark theorem 3.12 that we get the desired triplet $\{H_1, \Gamma, \Omega_1\}$, *which is the central point of the construction of a conservative extension for a time dispersive system satisfying the power dissipation condition (4.4) and (4.8).*

We may present a more constructive picture of the triplet the triplet $\{H_1, \Gamma, \Omega_1\}$ under the additional assumptions that

$$\operatorname{Re}\hat{a}(\sigma + i0) = \operatorname*{stlim}_{\eta \to 0} \operatorname{Re}\hat{a}(\sigma + i\eta) \tag{4.14}$$

exists for almost every σ, and

$$\lim_{\eta \to 0} \int_I \left\| \operatorname{Re}\hat{a}(\sigma + i\eta) - \operatorname{Re}\hat{a}(\sigma + i0) \right\| d\sigma = 0 \tag{4.15}$$

for any finite interval I. With these assumptions, the Stieltjes–Inversion formula—discussed in Appendix A.1 below—provides the following explicit formula for dK:

$$dK(\sigma) = \frac{1}{\pi} \operatorname{Re}\hat{a}(\sigma + i0)\, d\sigma. \tag{4.16}$$

Let us define $N(\sigma) = \pi^{-1}\operatorname{Re}\hat{a}(\sigma + i0)$, and note that $N(\sigma)$ is a non-negative operator for almost every σ. We choose the Hilbert space $H_1 = L^2(\mathbb{R}, H_0)$, which is the space of square integrable functions from $\mathbb{R}$ to H_0 with inner product

$$\langle w_1, w_2 \rangle_{H_1} = \int_{-\infty}^{\infty} \langle w_1(\sigma), w_2(\sigma) \rangle_{H_0}\, d\sigma. \tag{4.17}$$

Choosing for Ω_1 the operator

$$\Omega_1 w(\sigma) = \sigma w(\sigma), \tag{4.18}$$

and for Γ the map

$$\Gamma w = \int_{-\infty}^{\infty} \sqrt{N(\sigma)}\, w(\sigma)\, d\sigma, \tag{4.19}$$

it is easy to see that

$$\hat{a}(\zeta) = i \int_{-\infty}^{\infty} \frac{1}{\zeta - \sigma} N(\sigma)\, d\sigma = i\Gamma \frac{1}{\zeta - \Omega_1} \Gamma^{\dagger}, \tag{4.20}$$

which is the desired representation.

The above representation on $H_1 = L^2(\mathbb{R}, H_0)$ may not be minimal if, for σ from a set of positive measure, $N(\sigma)$ has a non-trivial kernel. In that case it is more useful to consider the Hilbert space $H_1 = L^2(dK)$ defined to be the space

$$\left\{ w \colon \mathbb{R} \to H_0 \text{ measurable } \middle| \int_{-\infty}^{\infty} \langle w(\sigma), N(\sigma) w(\sigma) \rangle \, d\sigma < \infty \right\}, \quad (4.21)$$

modulo null functions with $N(\sigma) w(\sigma) = 0$ for almost every σ. In fact the proof of the Naimark theorem (without assumptions (4.14)–(4.15)) proceeds by constructing the space $L^2(dK)$. A sketch of this construction is given in Appendix A.2.

Finally, having found the triplet $\{H_1, \Gamma, \Omega_1\}$, we construct the conservative extension as the system of equations (4.2)–(4.3). Using the matrix form (4.7) of the conservative system (4.2)–(4.3) we also get the following representation for the admittance operator

$$\mathfrak{A}(\zeta) = iT(\zeta\mathcal{M} - \mathcal{A})^{-1} T^{\dagger} = \left\{\zeta m - \left[A - i\hat{a}(\zeta)\right]\right\}^{-1}. \quad (4.22)$$

The above construction of the triplet $\{H_1, \Gamma, \Omega_1\}$, including the Hilbert space H_1 of "hidden variables", is essentially independent of the operator A which describes the internal dynamics of the "observable" variables.

4.2. Admittance Operator Scheme

Suppose the original time dispersive system is given by its admittance operator $\mathfrak{A}(\zeta)$, acting in the Hilbert space of "observable" variables H_0 by the following equation:

$$\hat{v}(\zeta) = \mathfrak{A}(\zeta) \hat{f}(\zeta), \quad \zeta = \omega + i\eta, \quad \eta = \mathrm{Im}\zeta > 0, \quad (4.23)$$

relating the generalized velocity $\hat{v}(\zeta)$ and the generalized force $\hat{f}(\zeta)$ in the complex frequency domain. For real $\zeta = \omega$ equation (4.23) reduces to the familiar real frequency form

$$\hat{v}(\omega) = \mathfrak{A}(\omega) \hat{f}(\omega). \quad (4.24)$$

We assume that the Hilbert space H_0 the admittance operator $\mathfrak{A}(\zeta)$ are already reduced in the sense of Definition 3.18, i.e. $H_0 = \tilde{H}_0$ and

$\mathfrak{A}_{\tilde{H}_0}(\zeta) = \mathfrak{A}(\zeta)$, and that $\mathfrak{A}(\zeta)$ satisfies the power dissipation and the growth conditions

$$\operatorname{Re}\mathfrak{A}(\zeta) \geqslant 0 \quad \text{for Im } \zeta > 0, \ \limsup_{\eta \to +\infty} \eta \, \|\mathfrak{A}(i\eta)\| < \infty. \tag{4.25}$$

We seek a conservative, non time-dispersive extension, of Equation (4.23) via the following representation for the admittance operator

$$\mathfrak{A}(\zeta) = iT\,(\zeta\mathcal{M} - \mathcal{A})^{-1}\, T^{\dagger} \tag{4.26}$$

with T an isometric truncation from a Hilbert space $\mathcal{H}$ to H_0, $\mathcal{M} \geqslant \delta I_{\mathcal{H}}$, $\delta > 0$ on $\mathcal{H}$, and $\mathcal{A}$ self-adjoint in $\mathcal{H}$, corresponding to the following evolution equation for v,

$$\mathcal{M}\dot{V}(t) = -i\mathcal{A}V(t) + T^{\dagger} f(t), \quad v(t) = TV(t). \tag{4.27}$$

To construct the representation (4.26), we first construct the Hilbert space $\mathcal{H}$ and the representation

$$\mathfrak{A}(\zeta) = i\Gamma_{\mathfrak{A}}(\zeta\mathcal{I} - \Omega_{\mathfrak{A}})^{-1}\,\Gamma_{\mathfrak{A}}^{\dagger}, \tag{4.28}$$

following the argument which led to (4.9) for the friction function in the previous section—applying Theorem 3.19 to $G(\zeta) = i\mathfrak{A}(\zeta)$.

Having obtained (4.28), we use Theorem 3.20 to get the following polar decomposition

$$\Gamma_{\mathfrak{A}} = U m_{\mathfrak{A}}^{-\frac{1}{2}} P_{H_0'} \quad \text{where } U\colon H_0' \to H_0 \text{ in unitary,} \tag{4.29}$$
$$m_{\mathfrak{A}} = U^{-1}\left[\Gamma_{\mathfrak{A}}\Gamma_{\mathfrak{A}}^{\dagger}\right]^{-1} U \geqslant \left\|\Gamma_{\mathfrak{A}}\Gamma_{\mathfrak{A}}^{\dagger}\right\|^{-1} I_{H_0'}, \quad \Gamma_{\mathfrak{A}}\Gamma_{\mathfrak{A}}^{\dagger} = \lim_{\eta \to +\infty} i\eta\mathfrak{A}(i\eta) > 0.$$

Then we introduce the mass operator

$$\mathcal{M} = m_{\mathfrak{A}} \oplus m_1 \quad \text{where } m_1\colon H_1 \to H_1 \geqslant \delta I_{H_1}, \ \delta > 0 \tag{4.30}$$

with m_1 being chosen as we please. In view of (4.29)

$$\Gamma_{\mathfrak{A}} = U P_{H_0'} \mathcal{M}^{-\frac{1}{2}} = U P_{H_0'}\left(m_{\mathfrak{A}}^{-\frac{1}{2}} \oplus m_1^{-\frac{1}{2}}\right) \tag{4.31}$$

and we get the desired representation

$$\mathfrak{A}(\zeta) = iT(\zeta\mathcal{M} - \mathcal{A})^{-1} T^{\dagger},$$
$$\text{where } T = UP_{H_0'} \quad \text{and} \quad \mathcal{A} = \mathcal{M}^{\frac{1}{2}} \Omega_{\mathfrak{A}} \mathcal{M}^{\frac{1}{2}}. \tag{4.32}$$

Note that the Hilbert space of "hidden variables" is $H_1 = \mathcal{H} \ominus H_0'$, which is $\ker \Gamma_{\mathfrak{A}}$, i.e.

$$H_1 = \{\psi \in \mathcal{H} : \Gamma_{\mathfrak{A}} \psi = 0\}. \tag{4.33}$$

5. EXAMPLES OF THE CONSTRUCTION OF CONSERVATIVE SYSTEMS

In this section we apply the general scheme for the construction of conservative extensions, as described in Sections 3 and 4, to a few well known classical systems: a damped oscillator, a general scalar dispersive dissipative system, and a classical dielectric medium. Related constructions have appeared elsewhere in the literature, e.g., Lamb's representation of a damped oscillator as a mass attached to an infinitely long tense string[9]. Interesting examples and very detailed studies of relations between admittance operators and spectral measures for loaded strings (as described by Krein–Feller operators) are offered in the second paper in ref. 7, wherein dispersion is introduced via boundary conditions.

5.1. Damped Oscillator

We consider a damped oscillator with mass $m_o > 0$, real frequency Ω_o, and friction coefficient $\gamma_o > 0$, described by a complex variable v which evolves according to the following equation:

$$m_o \partial_t v = -i m_o \Omega_o v - \gamma_o v + f(t), \tag{5.1}$$

where $f(t)$, the external force, is a complex-valued function. Evidently, Equation (5.1) is a particular case of a system of the form (1.1) with

$$H_0 = \mathbb{C}, \quad A = m_o \Omega_o, \quad a(t) = a_o(t) = \gamma_o \delta(t) \quad \text{and} \quad \hat{a}_o(\zeta) = \gamma_o \quad \text{for } \operatorname{Im} \zeta \geqslant 0. \tag{5.2}$$

Following the friction function scheme described in Section 3, we look at Equation (5.1) as a consequence of a larger conservative system with hidden degrees of freedom of the form (4.2)–(4.3), and proceed with the construction of the triplet $\{H_1, \Omega_1, \Gamma\}$ based on Theorem 3.16. Notice that $G(\zeta) = i\hat{a}_o(\zeta) = i\gamma_o$ evidently satisfies the hypothesis of Theorem 3.16. Hence, we must have

$$\gamma_o = i \lim_{R\to\infty} \Gamma \frac{1}{\zeta I_{H_1} - \Omega_1} \left(\Gamma \frac{R^2}{\Omega_1^2 + R^2 I_{H_1}} \right)^{\dagger}, \quad \operatorname{Im} \zeta \geqslant 0. \tag{5.3}$$

The discussion following Equation (4.16)—in particular Equation (4.20) with $N(\sigma) = \gamma_0/\pi$—suggests that we take $H_1 = L^2(\mathbb{R})$,

$$\Omega_1 \psi(\sigma) = \sigma \psi(\sigma), \quad \sigma \in \mathbb{R}, \quad \psi(\sigma) \in L_2(\mathbb{R}) \tag{5.4}$$

and

$$\Gamma \psi = \sqrt{\frac{\gamma_o}{\pi}} \lim_{R\to\infty} \int_{-R}^{R} \psi(\sigma) = \sqrt{\frac{\gamma_o}{\pi}} \langle \mathbf{1}, \psi(\sigma) \rangle, \quad \text{where } \mathbf{1} = \mathbf{1}(\sigma) = 1, \quad \sigma \in \mathbb{R}. \tag{5.5}$$

Although Γ is not bounded, we note that Γ is an Ω_1-bounded map from $\mathcal{D}(\Omega_1) \to \mathbb{C}$, since

$$|\Gamma \psi| \leqslant \sqrt{\frac{\gamma_o}{\pi}} \lim_{R\to\infty} \int_{-R}^{R} \left| \frac{1}{\sigma + i} \right| |(\sigma + i)\psi(\sigma)| \, d\sigma \leqslant \sqrt{\gamma_o} \, \|(\Omega_1 + i)\psi\|. \tag{5.6}$$

Formally,

$$\left(\Gamma^{\dagger} v\right)(\sigma) = \sqrt{\frac{\gamma_o}{\pi}} v, \quad \sigma \in \mathbb{R}. \tag{5.7}$$

The generator of the dynamics in the conservative system is the self-adjoint operator $\mathcal{A}$ acting in the Hilbert space $\mathcal{H} = \mathbb{C} \oplus L_2(\mathbb{R}, d\sigma)$, defined by

$$\begin{aligned} \mathcal{A} \begin{bmatrix} v \\ \psi(\sigma) \end{bmatrix} &= \begin{bmatrix} m_o \Omega_o & \sqrt{\frac{\gamma_o}{\pi}} \langle \mathbf{1}, \cdot \rangle \\ \sqrt{\frac{\gamma_o}{\pi}} \mathbf{1} & \sigma \end{bmatrix} \begin{bmatrix} v \\ \psi(\sigma) \end{bmatrix} \\ &= \begin{bmatrix} m_o \Omega_o v + \sqrt{\frac{\gamma_o}{\pi}} \langle \mathbf{1}, \psi(\sigma) \rangle \\ \sqrt{\frac{\gamma_o}{\pi}} v + \sigma \psi(\sigma) \end{bmatrix}, \quad \sigma \in \mathbb{R} \end{aligned} \tag{5.8}$$

with the domain

$$D(\mathcal{A}) = \left\{ \begin{bmatrix} v \\ \psi(\sigma) \end{bmatrix}, \ \sigma \in \mathbb{R} : \sqrt{\frac{\gamma_o}{\pi}} v + \sigma \psi(\sigma) \in L_2(\mathbb{R}, d\sigma) \right\}. \tag{5.9}$$

One can show that for any vector in $D(\mathcal{A})$ the expression $\langle \mathbf{1}, \psi(\sigma) \rangle$ is well defined as the following limit

$$\langle \mathbf{1}, \psi(\sigma) \rangle := \lim_{R \to \infty} \int_{-R}^{R} \psi(\sigma) \, d\sigma \tag{5.10}$$

and by Proposition 2.2, we see that $\mathcal{A}$ is self-adjoint.

Finally the mass operator here takes the form

$$\mathcal{M} = \begin{bmatrix} m_o & 0 \\ 0 & I_{H_1} \end{bmatrix} \tag{5.11}$$

and the desired extended conservative system for the damped oscillator is

$$\begin{aligned} m_o \partial_t v &= -i m_o \Omega_o v - i \sqrt{\frac{\gamma_o}{\pi}} \int_{-\infty}^{\infty} \psi(\sigma) \, d\sigma + f(t), \\ \partial_t \psi(\sigma) &= -i \sigma \psi(\sigma) + i \sqrt{\frac{\gamma_o}{\pi}} v \end{aligned} \tag{5.12}$$

for $(v, \psi) \in \mathcal{H} = \mathbb{C} \oplus L_2(\mathbb{R}, d\sigma)$.

5.2. General Scalar Dispersive Dissipative System

A number of classical dispersive systems, including homogeneous dielectrics and an oscillator with a retarded friction, can be described by a scalar complex variable v governed by an evolution equation

$$m_s \partial_t v(t) = -i m_s \Omega_s v(t) - \int_0^{\infty} a_s(\tau) v(t - \tau) \, d\tau + f(t) \tag{5.13}$$

with $a_s(\tau)$ complex-valued functions, m_s a positive number and Ω_s a real number. The function f is the external force. The scalar friction function $a_s(t)$ is assumed to satisfy Condition 3.5 and the power dissipation condition (1.19), which is to say it is a positive definite function as in the classical Bochner's theorem. The system described (5.13) has, according to (2.44), the following admittance form

$$\hat{v}(\zeta) = \mathfrak{A}_{\mathrm{s}}(\zeta)\,\hat{f}(\zeta), \quad \mathfrak{A}_{\mathrm{s}}(\zeta) = \left\{m_{\mathrm{s}}(\zeta - \Omega_s) + i\hat{a}_s(\zeta)\right\}^{-1}, \;\; \mathrm{Im}\;\zeta > 0. \tag{5.14}$$

Let us consider here a general scalar dispersive dissipative system given in admittance operator form

$$\hat{v}(\zeta) = \mathfrak{A}_{\mathrm{s}}(\zeta)\,\hat{f}(\zeta), \quad \zeta = \omega + i\eta, \;\; \eta = \mathrm{Im}\;\zeta > 0, \tag{5.15}$$

where the scalar admittance operator $\mathfrak{A}_{\mathrm{s}}(\zeta)$ satisfies the power dissipation condition

$$\mathrm{Re}\,\mathfrak{A}_{\mathrm{s}}(\zeta) \geqslant 0, \quad \zeta = \omega + i\eta, \;\; \eta = \mathrm{Im}\;\zeta > 0 \tag{5.16}$$

and

$$\limsup_{\eta\to\infty} \eta\,|\mathfrak{A}_{\mathrm{s}}(i\eta)| < \infty \tag{5.17}$$

and $\mathfrak{A}_{\mathrm{s}}(\zeta) \neq 0$ at least for one ζ.

To find a conservative extension for (5.15), we use the admittance operator scheme from Section 4. In this case $H_0 = \mathbb{C}$ and we first seek a Hilbert space $\mathcal{H}$ and a self-adjoint operator $\Omega_{\mathfrak{A}_{\mathrm{s}}}$ in it such that

$$\begin{aligned} \mathfrak{A}_{\mathrm{s}}(\zeta) &= i\Gamma_{\mathfrak{A}_{\mathrm{s}}}\left(\zeta I - \Omega_{\mathfrak{A}_{\mathrm{s}}}\right)^{-1}\Gamma_{\mathfrak{A}_{\mathrm{s}}}^{\dagger} \\ &= -i\int_{-\infty}^{\infty} \frac{\Gamma_{\mathfrak{A}_{\mathrm{s}}} E_{\Omega_{\mathfrak{A}_{\mathrm{s}}}}(d\sigma)\,\Gamma_{\mathfrak{A}_{\mathrm{s}}}^{\dagger}}{\sigma - \zeta}, \quad \mathrm{Im}\;\zeta \geqslant 0. \end{aligned} \tag{5.18}$$

We construct the spectral representation (5.18) as follows. In view of conditions (5.16) and (5.17), the classical Nevanlinna theorem 3.8 gives a non-negative scalar measure $dN_{\mathfrak{A}_{\mathrm{s}}}(\sigma)$ satisfying

$$i\mathfrak{A}_{\mathrm{s}}(\zeta) = \int_{-\infty}^{\infty} \frac{dN_{\mathfrak{A}_{\mathrm{s}}}(\sigma)}{\sigma - \zeta}, \quad \mathrm{Im}\;\zeta \geqslant 0, \quad dN_{\mathfrak{A}_{\mathrm{s}}}(\sigma) \geqslant 0. \tag{5.19}$$

Let us define $m_{\mathfrak{A}_{\mathrm{s}}}$ by

$$\frac{1}{m_{\mathfrak{A}_{\mathrm{s}}}} = \int_{-\infty}^{\infty} dN_{\mathfrak{A}_{\mathrm{s}}}(\sigma) = \lim_{\eta\to\infty} \eta\mathfrak{A}_{\mathrm{s}}(i\eta) \tag{5.20}$$

and note that if $\mathfrak{A}_{\mathrm{s}}$ is given by (5.14), $m_{\mathfrak{A}_{\mathrm{s}}} = m_{\mathrm{s}}$.

We set

$$\mathcal{H}=L_2\left(dN_{\mathfrak{A}_s}(\sigma),\,\mathbb{C}\right),\quad H_0'=\{\psi\in\mathcal{H}:\psi(\sigma)=v\in\mathbb{C},\ \sigma\in\mathbb{R}\}.\quad (5.21)$$

In other words, H_0' *is defined as a set of constant functions of* σ,which is, evidently, unitarily equivalent to the set of complex numbers $\mathcal{C}$ through the following mapping

$$U\colon H_0'\to\mathbb{C},\ \ U(v\mathbf{1})=\frac{v}{\sqrt{m_{\mathfrak{A}_s}}},\quad \text{where } \mathbf{1}=\mathbf{1}(\sigma)=1\quad \text{for } \sigma\in\mathbb{R}.\ (5.22)$$

We also define

$$\Omega_{\mathfrak{A}_s}\psi(\sigma)=\sigma\psi(\sigma),\ \ \psi\in\mathcal{H},\quad \left[\Gamma_{\mathfrak{A}_s}\psi\right]=(\mathbf{1},\psi)=\int\psi(\sigma)\,dN_{\mathfrak{A}_s}(\sigma)\in\mathbb{C} \quad (5.23)$$

implying

$$\Gamma_{\mathfrak{A}_s}^{\dagger}v=v\mathbf{1},\quad v\in\mathbb{C}. \quad (5.24)$$

The representation (5.18) readily follows from the definitions (5.21)–(5.24). Also, from (5.23) and (5.24) it follows that

$$\Gamma_{\mathfrak{A}_s}\Gamma_{\mathfrak{A}_s}^{\dagger}=\int_{-\infty}^{\infty}dN_{\mathfrak{A}_s}(\sigma)=\frac{1}{m_{\mathfrak{A}_s}}\colon\mathbb{C}\to\mathbb{C}. \quad (5.25)$$

Observe that the orthogonal projection $P_{H_0'}$ is

$$\left[P_{H_0'}\psi\right](\sigma)=\left[m_{\mathfrak{A}_s}\int_{-\infty}^{\infty}\psi(\sigma)\,dN_{\mathfrak{A}_s}(\sigma)\right]\mathbf{1},\quad \sigma\in\mathbb{R}. \quad (5.26)$$

We may choose the mass operator $\mathcal{M}$ to be just the scalar operator $\mathcal{M}=m_{\mathfrak{A}_s}I_{\mathcal{H}}$, and, consequently, get the desired components of an extended conservative system

$$\mathcal{H}=L_2\left(dN_{\mathfrak{A}_s}(\sigma),\mathbb{C}\right),\quad \mathcal{M}=m_{\mathfrak{A}_s}I_{\mathcal{H}},\quad \mathcal{A}\psi(\sigma)=m_{\mathfrak{A}_s}\sigma\psi(\sigma),\quad T=UP_{H_0'}, \quad (5.27)$$

where U is defined by (5.22).

The measure $dN_{\mathfrak{A}}(\sigma)$ can be recovered from Stieltjes' formula—see Equation (A.3) below

$$\int_{-\infty}^{\infty} f(\sigma) dN_{\mathfrak{A}_s}(\sigma) = \lim_{\eta \to +0} \frac{1}{\pi} \int_{-\infty}^{\infty} f(\sigma) \operatorname{Re}\{\mathfrak{A}_s(\sigma + i\eta)\}\, d\sigma, \quad \text{for } f \in C_0(\mathbb{R}). \tag{5.28}$$

In particular, if $dN_{\mathfrak{A}_s}(\sigma)$ has a density $n_{\mathfrak{A}_s}(\sigma)$ with respect to the Lebesgue measure, i.e.

$$dN_{\mathfrak{A}_s}(\sigma) = n_{\mathfrak{A}_s}(\sigma)\, d\sigma, \tag{5.29}$$

then the density $n_{\mathfrak{A}_s}(\sigma)$ is the following pointwise limit

$$n_{\mathfrak{A}_s}(\sigma) = \lim_{\eta \to 0} \frac{1}{\pi} \operatorname{Re}\{\mathfrak{A}_s(\sigma + i\eta)\}, \tag{5.30}$$

which, in view of (5.20), satisfies

$$n_{\mathfrak{A}_s}(\sigma) \geqslant 0 \quad \text{and} \quad \int_{-\infty}^{\infty} n_{\mathfrak{A}_s}(\sigma)\, d\sigma = \lim_{\eta \to \infty} \eta \mathfrak{A}_s(i\eta) = \frac{1}{m_{\mathfrak{A}_s}}. \tag{5.31}$$

When the measure $dN_{\mathfrak{A}_s}(\sigma)$ has a density $n_{\mathfrak{A}_s}(\sigma)$, there is a slightly different description based on the Hilbert space $\mathcal{H} = L_2(\mathbb{R})$, namely:

$$\Omega_{\mathfrak{A}_s} \psi(\sigma) = \sigma \psi(\sigma), \quad \psi \in L_2(\mathbb{R}), \tag{5.32}$$

$$H_0' = \left\{ \psi \in L_2(\mathbb{C}) : \psi(\sigma) = v \sqrt{n_{\mathfrak{A}_s}(\sigma)},\ v \in \mathbb{C} \right\}, \tag{5.33}$$

$$\left[\Gamma_{\mathfrak{A}_s} \psi\right](\sigma) = \int_{-\infty}^{\infty} \sqrt{n_{\mathfrak{A}_s}(\sigma)} \psi(\sigma)\, d\sigma, \ \left[\Gamma_{\mathfrak{A}_s}^{\dagger} v\right](\sigma) = v \sqrt{n_{\mathfrak{A}_s}(\sigma)}. \tag{5.34}$$

Then we get

$$U\colon H_0' \to \mathbb{C}, \quad U\left(v \sqrt{n_{\mathfrak{A}_s}(\sigma)}\right) = \frac{v}{m_{\mathfrak{A}_s}}, \tag{5.35}$$

$$\left[P_{H_0'} \psi\right](\sigma) = \left[m_{\mathfrak{A}_s} \int_{-\infty}^{\infty} \sqrt{n_{\mathfrak{A}_s}(\sigma')} \psi\left(\sigma'\right) d\sigma'\right] \sqrt{n_{\mathfrak{A}_s}(\sigma)}, \qquad \sigma \in \mathbb{R}, \tag{5.36}$$

and, consequently,

$$\mathcal{H} = L_2(\mathbb{C}), \quad \mathcal{M} = m_{\mathfrak{A}_s} I_{\mathcal{H}}, \quad \mathcal{A}\psi(\sigma) = m_{\mathfrak{A}_s} \sigma \psi(\sigma), \ T = U P_{H_0'}. \tag{5.37}$$

Finally, the evolution of the extended conservative system, as described by its state $\psi(t,\sigma)$, is governed by the following equations

$$m_{\mathfrak{A}_s}\partial_t\psi(t,\sigma) = -im_{\mathfrak{A}_s}\sigma\psi(t,\sigma) + f(t)\sqrt{m_{\mathfrak{A}_s}n_{\mathfrak{A}_s}(\sigma)}, \quad \psi(t,\sigma)\in L_2(\mathbb{C}) \tag{5.38}$$

with $n_{\mathfrak{A}_s}(\sigma) = \lim_{\eta\to 0}\frac{1}{\pi}\mathrm{Re}\{\mathfrak{A}_s(\sigma+i\eta)\} \geqslant 0$ and $f(t)$ the external force. *The state $v(t)$ of the scalar dispersive system (5.13) is represented by the following linear functional of $\psi(t,\sigma)$*

$$v(t) = T\psi(t) = \int_{-\infty}^{\infty}\sqrt{m_{\mathfrak{A}_s}n_{\mathfrak{A}_s}(\sigma)}\psi(t,\sigma)\,d\sigma. \tag{5.39}$$

It is of interest to note that under this construction, the extended system described by ψ is always governed by the canonical evolution equation (5.38) in the Hilbert space $\mathcal{H} = L_2(\mathbb{C})$ with generator $\mathcal{A}\psi(\sigma) = \sigma\psi(\sigma)$. Consequently, the only feature which distinguishes different scalar dispersive systems is the mass $m_{\mathfrak{A}_s}$ and the "observable variable" $v(t)$ obtained by projecting onto a one-dimensional Hilbert space H_0 spanned by the vector $\sqrt{m_{\mathfrak{A}_s}n_{\mathfrak{A}_s}(\sigma)}$ in $L_2(\mathbb{C})$. Observe that the external force $f(t)\sqrt{m_{\mathfrak{A}_s}n_{\mathfrak{A}_s}(\sigma)}$ is in the space H_0'.

5.3. Maxwell Equations for Lossy and Dispersive media

In this section we construct a conservative extension of the Maxwell equations for a homogeneous, lossy and dispersive medium following the friction-admittance scheme from Section 3.3.

The classical Maxwell equations for a homogeneous, lossy and dispersive medium are (ref. 2, Section 1.1)

$$\nabla\times\mathbf{E}(\mathbf{r},t) = -\partial_t\mathbf{B}(\mathbf{r},t) - 4\pi\mathbf{J}_B(\mathbf{r},t)\,, \quad \nabla\cdot\mathbf{H}(\mathbf{r},t) = 0\,, \tag{5.40}$$

$$\nabla\times\mathbf{H}(\mathbf{r},t) = \partial_t\mathbf{D}(\mathbf{r},t) + 4\pi\mathbf{J}(\mathbf{r},t)\,, \quad \nabla\cdot\mathbf{D}(\mathbf{r},t) = 0\,, \tag{5.41}$$

where $\mathbf{H}$, $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{D}$ are respectively the magnetic and electric fields, and magnetic and electric inductions, and $\mathbf{J}$ and $\mathbf{J}_B$ are respectively the external electric and magnetic currents. For simplicity, we consider here a non-magnetic medium, which amounts to taking

$$\mathbf{H} = \mathbf{B} \quad \text{and} \quad \mathbf{J}_B = 0 \tag{5.42}$$

in units such that $\mu_0 = 1$. We also assume there are no free charges, which is the assumption that the current $\mathbf{J}(\mathbf{r}, t)$ is divergence free

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0. \tag{5.43}$$

The dispersive properties of the medium come through the material (constitutive) relations which, in the simplest case of a homogeneous and isotropic medium, take the form

$$\mathbf{D}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) + 4\pi \mathbf{P}(\mathbf{r}, t), \text{ where } \mathbf{P}(\mathbf{r}, t) = \int_0^\infty \chi(\tau) \mathbf{E}(\mathbf{r}, t - \tau)\, d\tau, \tag{5.44}$$

and $\chi(\tau)$ is the scalar-valued *response (aftereffect) function* (ref. 3, chapter 2; ref. 8, Section 3). In the frequency domain the relation (5.44) between the polarization $\mathbf{P}(\mathbf{r}, t)$ and the electric field $\mathbf{E}(\mathbf{r}, t)$ becomes

$$\hat{\mathbf{P}}(\mathbf{r}, \omega) = \hat{\chi}(\omega) \hat{\mathbf{E}}(\mathbf{r}, \omega), \quad \hat{\chi}(\omega) = \int_0^\infty \chi(t)\, e^{i\omega t}\, dt, \tag{5.45}$$

where $\hat{\chi}(\omega)$ is the so-called *frequency dependent electric susceptibility*, which is a scalar-valued function for the case we consider. Since the medium is homogeneous and isotropic, $\hat{\chi}$ does not depend on $\mathbf{r}$ and we have

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \nabla \cdot \mathbf{E}(\mathbf{r}, t) = \nabla \cdot \mathbf{P}(\mathbf{r}, t) = 0. \tag{5.46}$$

To construct a conservative extension we begin by recasting the Maxwell equations for the dispersive medium in the general form from Section 3. The "observable" variables v and the corresponding Hilbert space H_0 in this case are

$$v(t) = \begin{bmatrix} \mathbf{E}(\mathbf{r}, t) \\ \mathbf{B}(\mathbf{r}, t) \end{bmatrix} \in H_0, \quad \text{where } H_0 = \left\{ v \in L^2(\mathbb{C}^6) : \nabla \cdot \mathbf{E}(\mathbf{r}, t) = \nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \right\}. \tag{5.47}$$

In other words, H_0 consists of square-integrable 6-dimensional fields with the components $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ divergence free. Comparing equations

(5.40)–(5.46) with the general evolution equation (1.1) we set

$$m_{\mathrm{M}} = I_{H_0}, \quad A_{\mathrm{M}} = \begin{bmatrix} 0 & i\nabla^{\times} \\ -i\nabla^{\times} & 0 \end{bmatrix}, \quad a_{\mathrm{M}}(t) = a_{\mathrm{s}}(t) \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}, \text{ where}$$

$$a_{\mathrm{s}}(t) = 4\pi\, \partial_t \chi(t), \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.48}$$

or, in the complex frequency domain

$$\hat{a}_{\mathrm{M}}(\zeta) = \hat{a}_s(\zeta) \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{a}_{\mathrm{s}}(\zeta) = -4\pi i \zeta \hat{\chi}(\zeta), \quad \mathrm{Im}\ \zeta \geqslant 0. \tag{5.49}$$

The reduced form of $\hat{a}_{\mathrm{M}}(\zeta)$—as in Definition 3.18—is

$$\tilde{H}_0 = \left\{ v \in L^2(\mathbb{C}^6) : \nabla \cdot \mathbf{E}(\mathbf{r}, t) = 0, \ \mathbf{B}(\mathbf{r}, t) = 0 \right\},$$
$$\hat{a}_{\mathrm{M}, \tilde{H}_0}(\zeta)\, \mathbf{E}(\mathbf{r}) = \hat{a}_s(\zeta)\, \mathbf{E}(\mathbf{r}). \tag{5.50}$$

Notice that the friction function $a_{\mathrm{M}}(t)$ is related to the time derivative of the polarization $\partial_t \mathbf{P}$, represented by the scalar function $\zeta \hat{\chi}(\zeta)$ in the complex frequency domain. That is consistent with the physical fact that the work per unit time done by the electric field $\mathbf{E}$ to produce the polarization is given by $\partial_t \mathbf{P} \cdot \mathbf{E}$. In view of the simple structure of the operator friction function (5.49) the power dissipation condition turns here into the following condition for the scalar function $\hat{a}_s(\zeta) = -4\pi i \zeta \hat{\chi}(\zeta)$

$$\mathrm{Re}\hat{a}_{\mathrm{s}}(\zeta) = 4\pi\, \mathrm{Im}\left\{ \zeta \hat{\chi}(\zeta) \right\} \geqslant 0, \quad \mathrm{Im}\ \zeta \geqslant 0. \tag{5.51}$$

It is easy to see that the construction of a conservative extension is essentially reduced to the construction of the conservative extension for the scalar friction function $a_{\mathrm{s}}(t)$. Thus, a conservative extension for the electric polarization can be found using the results of the previous section for a general scalar dispersive dissipative system and we obtain, in particular, the evolution equations (5.38) and the representation (5.39).

For simplicity, suppose that the following limit exists for every real σ

$$n_{\hat{\chi}}(\sigma) = \lim_{\eta \to +0} \frac{1}{\pi} \mathrm{Re}\left\{ \hat{a}_{\mathrm{s}}(\sigma + i\eta) \right\} = \lim_{\eta \to +0} 4\mathrm{Im}\left\{ (\sigma + i\eta) \hat{\chi}(\sigma + i\eta) \right\} \tag{5.52}$$

and that

$$\lim_{\eta\to\infty}\eta\mathrm{Re}\left\{\hat{a}_{\mathrm{s}}(i\eta)\right\}=\lim_{\eta\to\infty}4\eta\mathrm{Im}\left\{(\sigma+i\eta)\,\hat{\chi}\,(\sigma+i\eta)\right\}<\infty, \tag{5.53}$$

conditions which are satisfied in many non-trivial examples. Then the desired *conservative extension of the original Maxwell equations (5.40)–(5.46) for a dispersive and dissipative dielectric medium* takes the form

$$\begin{aligned}
\partial_t \mathbf{H}(\mathbf{r},t) &= -\nabla\times\mathbf{E}(\mathbf{r},t)\,,\\
\partial_t \mathbf{E}(\mathbf{r},t) &= \nabla\times\mathbf{H}(\mathbf{r},t)\\
&\quad -\int_{-\infty}^{\infty}\sqrt{m_{\hat{\chi}}n_{\hat{\chi}}(\sigma)}\Psi(\mathbf{r},t,\sigma)\,d\sigma-4\pi\mathbf{J}(\mathbf{r},t)\,, \qquad (5.54)\\
m_{\hat{\chi}}\partial_t\Psi(\mathbf{r},t,\sigma) &= -im_{\hat{\chi}}\sigma\Psi(\mathbf{r},t,\sigma)+\sqrt{m_{\hat{\chi}}n_{\hat{\chi}}(\sigma)}\mathbf{E}(\mathbf{r},t)\,,
\end{aligned}$$

where

$$\begin{aligned}
&n_{\hat{\chi}}(\sigma)=-4\lim_{\eta\to+0}\mathrm{Im}\left\{(\sigma+i\eta)\,\hat{\chi}\,(\sigma+i\eta)\right\}\geqslant 0,\quad m_{\hat{\chi}}^{-1}=\int_{-\infty}^{\infty}n_{\hat{\chi}}(\sigma)\,d\sigma,\\
&4\pi\zeta\hat{\chi}(\zeta)=\int_{-\infty}^{\infty}\frac{n_{\hat{\chi}}(\sigma)}{\sigma-\zeta}\,d\sigma,\quad \mathrm{Im}\zeta>0,\ \ \Psi(\mathbf{r},t,\sigma)\in L_2\left(\mathbb{R};\mathbb{C}^3\right), \qquad (5.55)
\end{aligned}$$

and the fields $\mathbf{H}(\mathbf{r},t)$, $\mathbf{D}(\mathbf{r},t)$ are divergence free, i.e.

$$\begin{aligned}
&\nabla\cdot\mathbf{H}(\mathbf{r},t)=0,\\
&\nabla\cdot\mathbf{E}(\mathbf{r},t)+4\pi\int_{-\infty}^{t}\int_{-\infty}^{\infty}\sqrt{m_{\hat{\chi}}n_{\hat{\chi}}(\sigma)}\nabla\cdot\Psi(\mathbf{r},\tau,\sigma)\,d\sigma d\tau=0. \qquad (5.56)
\end{aligned}$$

The electric polarization $\mathbf{P}(\mathbf{r},t)$, its time derivative $\partial_t\mathbf{P}(\mathbf{r},t)$ and the electric induction $\mathbf{D}(\mathbf{r},t)$ are now defined by

$$\begin{aligned}
\mathbf{P}(\mathbf{r},t) &= \frac{1}{4\pi}\int_{-\infty}^{t}\int_{-\infty}^{\infty}\sqrt{m_{\hat{\chi}}n_{\hat{\chi}}(\sigma)}\Psi(\mathbf{r},\tau,\sigma)\,d\sigma d\tau,\\
\partial_t\mathbf{P}(\mathbf{r},t) &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\sqrt{m_{\hat{\chi}}n_{\hat{\chi}}(\sigma)}\Psi(\mathbf{r},t,\sigma)\,d\sigma, \qquad (5.57)\\
\mathbf{D}(\mathbf{r},t) &= \mathbf{E}(\mathbf{r},t)+\int_{-\infty}^{t}\int_{-\infty}^{\infty}\sqrt{m_{\hat{\chi}}n_{\hat{\chi}}(\sigma)}\Psi(\mathbf{r},\tau,\sigma)\,d\sigma d\tau
\end{aligned}$$

We note that according to (5.54) the vector $\frac{1}{4\pi}\int_{-\infty}^{t}\Psi(\mathbf{r},\tau,\sigma)\,d\tau$ in (5.57) evidently represents a time dependent microscopic dipole of mass

$m_{\hat{\chi}}$, localized at $\mathbf{r}$, which oscillates with natural frequency σ. The total polarization $\mathbf{P}(\mathbf{r},t)$ of the medium at point $\mathbf{r}$ is a superposition of a (continuum) number of microscopic dipoles localized at $\mathbf{r}$. The natural definition for the energy of the "hidden" medium is

$$\frac{1}{2}\int_{\mathbb{R}^3}\int_{-\infty}^{\infty}|\Psi(\mathbf{r},t,\sigma)|^2\,d\sigma d\mathbf{r}, \tag{5.58}$$

consistent with the evolution equation and the assumption that $\int_{\mathbb{R}^3}\mathbf{E}(\mathbf{r},t)\cdot\partial_t\mathbf{P}(\mathbf{r},t)d\mathbf{r}$ is the instantaneous rate of work done by the electromagnetic field.

In a forthcoming paper, we shall discuss in greater detail conservative models for dispersion in dielectric media—including scattering theory and inhomogeneous media—however, to complete the discussion here it may be useful to consider one physically relevant non-trivial example. For this purpose, let us take the so-called Lorentz medium in which (ref. 17, Section 3.5; ref. 3, Section 9.1)

$$\chi(t)=\chi_{\mathrm{L}}(t)=\omega_{\mathrm{p}}^2\exp\left\{-\frac{\gamma}{2}t\right\}\frac{\sin\xi t}{\xi},\quad \xi=\sqrt{\omega_0^2-\frac{\gamma^2}{4}},\ t\geqslant 0; \tag{5.59}$$

$$\hat{\chi}(\zeta)=\hat{\chi}_{\mathrm{L}}(\zeta)=\frac{\omega_{\mathrm{p}}^2}{\omega_0^2-\zeta^2-i\gamma\zeta} \tag{5.60}$$

with ω_{p}, ω_0 and γ positive parameters. Observe that

$$\operatorname{Im}\left\{\zeta\hat{\chi}_{\mathrm{L}}(\zeta)\right\}=\frac{\omega_{\mathrm{p}}^2\left[\eta\left(\omega_0^2+\omega^2+\eta^2+\gamma\eta\right)+\omega^2\gamma\right]}{\left(\omega_0^2-\omega^2+\eta^2+\gamma\eta\right)^2+\omega^2(2\eta+\gamma)^2},\quad \zeta=\omega+i\eta \tag{5.61}$$

and hence

$$\operatorname{Im}\left\{\zeta\hat{\chi}_{\mathrm{L}}(\zeta)\right\}\geqslant 0,\quad \operatorname{Im}\zeta=\eta\geqslant 0, \tag{5.62}$$

in full compliance with the power dissipation condition (5.51). In addition, from (5.30), (5.31) we see that

$$n_{\hat{\chi}_{\mathrm{L}}}(\sigma)=4\lim_{\eta\to+0}\operatorname{Im}\left\{(\sigma+i\eta)\,\hat{\chi}_{\mathrm{L}}(\sigma+i\eta)\right\}=4\frac{\omega_{\mathrm{p}}^2\sigma^2\gamma}{\left(\omega_0^2-\sigma^2\right)^2+\sigma^2\gamma^2} \tag{5.63}$$

exists pointwise, and

$$m_{\hat{\chi}_L}^{-1} = \int_{-\infty}^{\infty} n_{\hat{\chi}}(\sigma)\, d\sigma = \lim_{\eta \to +\infty} \eta^2 \hat{\chi}_L(i\eta) = 4\pi\omega_p^2. \tag{5.64}$$

Plugging in the above values $n_{\hat{\chi}_L}(\sigma)$ and $m_{\hat{\chi}_L}$ into extended Maxwell equations (5.54)–(5.57) we get the desired conservative description of the Lorentz medium.

6. DISSIPATION AND CONTINUITY OF THE SPECTRUM

It may seem startling that dissipation, i.e., losses, can arise when we truncate a *unitary* evolution. However, such results are familiar from the theory of unitary dilations of contractive semi-groups, (12) which is the theory of solutions to (1.1) for friction without retardation, i.e., $a(t) = \alpha_\infty \delta(t)$. Furthermore, the mechanism at work is physically very natural: there is energy transport to a large number of “invisible” degrees of freedom, i.e. to “heat.”

Mathematically, a rigorous analysis of losses in very large but finite systems is generally complicated by the fact that systems with a finite number of degrees of freedom eventually (perhaps after an extremely long time) return arbitrarily close to their starting configuration (Poincaré recurrence). However, if we consider an idealization in which there are infinitely many hidden degrees of freedom, resulting in infinite recurrence time, then the Poincaré recurrence may not occur and there is hope of describing losses in a cleaner and simpler way. *As it turns out, a sufficient condition for losses is strict positivity of* Re $\hat{a}(\zeta)$, *which implies absolute continuity of the spectral measure for the generator* $\mathcal{A}$ *of the dynamics of a conservative extension.*

To state a quite general condition, we use the notion of *non-tangential boundedness*. Given $\omega_0 \in \mathbb{R}$ let the cone of aperture $\theta \in (0, \pi)$ at ω_0, denoted $\Gamma_\theta(\omega_0)$, be the set

$$\Gamma_\theta(\omega_0) = \left\{ \omega + i\eta : \eta < 1 \text{ and } |\omega - \omega_0| < \eta \tan\frac{\theta}{2} \right\}. \tag{6.1}$$

A function $F\colon \mathbb{C}_+ \to \mathbb{C}$, with $\mathbb{C}_+ = \{\zeta : \mathrm{Im}\ \zeta > 0\}$, is said to be *non-tangentially bounded at* $\omega_0 \in \mathbb{R}$ if

$$\sup_{\zeta \in \Gamma_\theta(\omega_0)} |F(\omega)| < \infty \tag{6.2}$$

for some $\theta \in (0, \pi)$. We shall use the following well known theorem from harmonic analysis regarding boundary values of non-tangentially bounded harmonic functions, see e.g. ref. 16.

Theorem 6.1. Let $F : \mathbb{C}_+ \to \mathbb{C}$ be a harmonic function and suppose that F is non-tangentially bounded at every point of a set $E \subset \mathbb{R}$. Then

$$F(\omega + i0) = \lim_{\eta \to 0} F(\omega + i\eta) \tag{6.3}$$

exists for almost every $\omega \in E$.

Note: In fact, the so-called non-tangential limits $\lim_{\zeta \to \omega} F(\zeta)$, with ζ restricted to $\Gamma_\theta(\omega)$, exist at almost every ω, but we will not use this fact.

Our principle theorem on losses is the following.

Theorem 6.2. Suppose the mass operator m is strictly positive, $m \geqslant \delta > 0$, and the Laplace transform $\hat{a}(\zeta)$ of the friction function satisfies the following strengthened form of the power-dissipation condition (2.42): there are measurable functions $\theta \colon \mathbb{R} \to (0, \pi)$, $\gamma \colon \mathbb{R} \to (0, \infty)$, with $\gamma(\cdot) \in L^1_{\text{loc}}(\mathbb{R})$ such that for almost every $\omega \in \mathbb{R}$

$$\operatorname{Re}\big(v, \hat{a}(\zeta)\, v\big) \geqslant \frac{1}{\gamma(\omega)} \|v\|^2, \quad \text{for all} \quad \zeta \in \Gamma_{\theta(\omega)}(\omega) \quad \text{and} \quad v \in H_1. \tag{6.4}$$

Then for any compactly supported generalized force $f \in L^1_c(\mathbb{R}; H_0)$, the solution $v_f(t)$ to the evolution equation (1.1) vanishes in the large t limit:

$$\lim_{t \to \infty} \left\| v_f(t) \right\| = 0. \tag{6.5}$$

Remark 6.3. The condition (6.4) holds in particular if $\operatorname{Re} \hat{a}(\zeta)$ is strictly positive, $\operatorname{Re} \hat{a}(\zeta) \geqslant \delta I_{H_0}$ for all $\operatorname{Im} \zeta > 0$. For example, in the Lorentz medium we have $\hat{a}(\zeta) = \gamma + i\frac{\omega_o^2}{\zeta}$, so $\operatorname{Re} \hat{a}(\zeta) \geqslant \gamma I_{H_0}$.

Proof. Note that the admittance $\mathfrak{A}(\zeta) = i\left(\zeta m - A + i\hat{a}(\zeta)\right)^{-1}$ obeys

$$\begin{aligned} \operatorname{Re} \mathfrak{A}(\zeta) &= \mathfrak{A}(\zeta)\left[\operatorname{Im} \zeta m + \operatorname{Re} \hat{a}(\zeta)\right]\mathfrak{A}(\zeta)^\dagger \\ &\geqslant \frac{1}{\gamma(\omega)} \mathfrak{A}(\zeta)\mathfrak{A}(\zeta)^\dagger, \quad \text{for } \zeta \in \Gamma_{\theta(\omega)}(\omega). \end{aligned} \tag{6.6}$$

We conclude that

$$\|\mathfrak{A}(\zeta)\| \geqslant \|\text{Re } \mathfrak{A}(\zeta)\| \geqslant \frac{1}{\gamma(\omega)} \|\mathfrak{A}(\zeta)\|^2, \tag{6.7}$$

and thus

$$\|\mathfrak{A}(\zeta)\| \leqslant \gamma(\omega) \quad \text{for } \zeta \in \Gamma_{\theta(\omega)}(\omega). \tag{6.8}$$

Using Theorem 3.13, we find a conservative extension consisting of a self-adjoint operator $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and an isometric imbedding $T^\dagger \colon H_0 \to \mathcal{H}$ such that $\mathfrak{A}(\zeta) = i m^{-1/2} T (\zeta - \mathcal{A})^{-1} T^\dagger m^{-1/2}$. Furthermore the solution v_f to (1.1) with generalized force f is

$$v_f(t) = \int_{-\infty}^{t} m^{-1/2} T e^{i(s-t)\mathcal{A}} T^\dagger m^{-1/2} f(s)\, ds. \tag{6.9}$$

The theorem now follows from Theorem 6.4 below via dominated convergence. ▮

Theorem 6.4. Let $\mathcal{A}$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$, and let $T \colon \mathcal{H} \to H_0$ be a bounded map from $\mathcal{H}$ into a separable Hilbert space H_0. Suppose there are measurable functions $\gamma \colon \mathbb{R} \to (0, \infty)$ and $\theta \colon \mathbb{R} \to (0, \pi)$ with $\gamma(\cdot) \in L^1_{\text{loc}}(\mathbb{R})$ such that for almost every $\omega \in \mathbb{R}$

$$\sup_{\zeta \in \Gamma_{\theta(\omega)}(\omega)} \left\| \text{Im } T(\zeta - \mathcal{A})^{-1} T^\dagger \right\| \leqslant \gamma(\omega). \tag{6.10}$$

Then

$$\lim_{t \to \pm\infty} T e^{-it\mathcal{A}} T^\dagger v = 0 \tag{6.11}$$

for every $v \in H_0$.

Remark 6.5. (1) As will be clear from the proof, if $\gamma(\cdot) \in L^1$ (instead of L^1_{loc}), then Equation (6.11) holds in *norm*. (2) Equation (6.10) implies that the spectral measure associated to $\mathcal{A}$ and any $\psi \in \operatorname{ran} T^\dagger$ is purely absolutely continuous, with density

$$-\lim_{\eta \to +0} \langle \psi, \text{Im } (\omega + i\eta - \mathcal{A})^{-1} \psi \rangle. \tag{6.12}$$

Proof. We begin by showing that for each bounded open interval $I \subset \mathbb{R}$

$$\left\| T\, e^{-it\mathcal{A}} E_{\mathcal{A}}(I) T^{\dagger} \right\| \to 0 : \text{as} : t \to \pm\infty, \tag{6.13}$$

where $E_{\mathcal{A}}(\cdot)$ is the spectral resolution of $\mathcal{A}$.

Let us define

$$F(\zeta) = \operatorname{Im}\ T^{\dagger}(\zeta - \mathcal{A})^{-1} T \qquad \text{for } \operatorname{Im}\ \zeta \neq 0. \tag{6.14}$$

By the spectral theorem and the observation that the spectral measures of $\mathcal{A}$, $d\|E_{\mathcal{A}}(\lambda)\psi\|^2$, are purely absolutely continuous, we have

$$T e^{-it\mathcal{A}} E_{\mathcal{A}}(I) T^{\dagger} v = -\frac{1}{\pi} \lim_{\eta \downarrow 0} \int_I d\omega\, e^{-it\omega} F(\omega + i\eta)\, v \tag{6.15}$$

for each finite t and every $v \in H_0$.

Since H_0 is separable, it has a countable basis ϕ_j, $j = 1, \ldots,$ Using Theorem 6.1 we conclude that there is $I' \subset I$ with $m(I \backslash I') = 0$ such that

$$\lim_{\varepsilon \to 0} \langle \phi_i, F(\omega + i\eta) \phi_j \rangle \tag{6.16}$$

exists for every $\omega \in I'$ and every pair of basis vectors ϕ_i, ϕ_j. Let $\mathcal{S}$ denote the subspace of linear combinations of finitely many basis vectors. Given $\psi \in \mathcal{S}$,

$$\begin{aligned} \sum_{i=1}^{\infty} \lim_{\eta \to 0} |\langle \phi_i, F(\omega + i\eta)\psi \rangle|^2 &\leqslant \liminf_{\eta \to 0} \sum_{i=1}^{\infty} |\langle \phi_i, F(\omega + i\eta)\psi \rangle|^2 \\ &= \liminf_{\eta \to 0} \|F(\omega + i\eta)\psi\|^2 \leqslant \gamma(\omega)^2 \|\psi\|^2. \end{aligned} \tag{6.17}$$

We conclude that, for every $\omega \in I'$, the limit

$$\text{wk-}\lim_{\eta} F(\omega + i\eta)\psi = \sum \lim_{\eta} \langle \phi_i, F(\omega + i\eta)\psi \rangle \phi_i \tag{6.18}$$

exists, and the map

$$\psi \mapsto \sum_{i=1}^{\infty} \left(\lim_{\eta \to 0} \langle \phi_i, F(\omega + i\eta)\psi \rangle \right) \phi_i \tag{6.19}$$

is bounded from $\mathcal{S}$ into H_0. Since $\mathcal{S}$ is dense, this map may be extended to a unique bounded linear map $F(\omega+i0)$: $H_0 \to H_0$ with $\|F(\omega+i0)\| \leqslant \gamma(u)$. It is elementary to see that,

$$\langle \psi_1, F(\omega+i0)\psi_2\rangle = \lim_{\eta\to 0} \langle \psi_1, F(\omega+i\eta)\psi_2\rangle \quad \text{for any } \psi_1, \psi_2 \in H_0, \tag{6.20}$$

i.e.

$$F(\omega+i0) = \operatorname*{wk-lim}_{\eta} F(\omega+i\eta) \text{ for } \omega \in I'. \tag{6.21}$$

By dominated convergence we find from (6.15)

$$T\, e^{-it\mathcal{A}} E_{\mathcal{A}}(I) T^{\dagger} v = -\frac{1}{\pi}\int_I d\omega e^{-it\omega}\, F(\omega+i0)\, v \tag{6.22}$$

for every $v \in H_0$. Since $\|F(\omega+i0)\| \leqslant \gamma(\omega) \in L^1(I)$, we have

$$\lim_{t\to\pm\infty} \left\| T e^{-it\mathcal{A}} E_{\mathcal{A}}(I) T^{\dagger} \right\| = 0 \tag{6.23}$$

by the Riemann–Lebesgue lemma—the extension of this result to operator valued functions is elementary.

To complete the proof, we note that given $\varepsilon > 0$ and $\psi \in H_0$ we can find a finite interval $I_\varepsilon \subset \mathbb{R}$ such that $\left\|(1 - E_{\mathcal{A}}(I_\varepsilon))T^{\dagger}\psi\right\| \leqslant \varepsilon$. Thus

$$\limsup_{t\to\pm\infty} \left\| T e^{-it\mathcal{A}} T^{\dagger}\psi \right\| \leqslant \limsup_{t\to\pm\infty} \left\| T e^{-it\mathcal{A}} E_{\mathcal{A}}(I_\varepsilon) T^{\dagger}\psi \right\| + \varepsilon = \varepsilon. \tag{6.24}$$

Since ε is arbitrary, we see that (6.11) holds. ▮

7. OPERATOR VERSIONS OF CLASSICAL SPECTRAL THEOREMS

In this section we discuss the proofs of operator versions of Bochner's Theorem 3.1—Theorems 3.2 and 3.6 above—and operator versions of the Herglotz–Nevanlinna Theorems 3.9, 3.8—Theorem 3.13 and 3.16. For properties of operator-valued functions holomorphic in a half-plane and their boundary values see ref. 15.

7.1. Bochner's Theorem

The two operator valued generalizations of Bochner's theorem stated above—Theorems 3.2 and 3.6—are combined in the following statement.

Theorem 7.1. The friction function $a_e(t) = 2\alpha_\infty \delta(t) + \alpha_e(t)$, $-\infty < t < \infty$, with $\alpha_e(t)$ a strongly continuous $\mathcal{B}(H_0)$ valued function, and α_∞ a bounded non-negative operator is representable as

$$a_e(t) = \operatorname*{Dlim}_{R\to\infty} \Gamma\, e^{-it\Omega_1} \left(\Gamma\, \Phi_R^2\right)^\dagger, \quad \Phi_R^2 = \left(\frac{\Omega_1^2}{R^2} + I_{H_1}\right)^{-1} \tag{7.1}$$

with Ω_1 a self-adjoint operator on H_1 and $\Gamma\colon D(\Omega_1) \to H_0$ a Ω_1-bounded linear map, if and only if $a(t)$ satisfies the dissipation condition (1.19) for every continuous H_0 valued function $v(t)$ with compact support. The operator Γ is bounded if and only if $\alpha_\infty = 0$, in which case

$$a_e(t) = \Gamma e^{-it\Omega_1} \Gamma^\dagger. \tag{7.2}$$

If the space H_1 is minimal—in the sense that

$$\left\langle (\Gamma f(\Omega_1))^\dagger v:\ f \in C_c(\mathbb{R}), v \in H_0 \right\rangle \tag{7.3}$$

is dense in H_1—then the triplet $\{H_1, \Omega_1, \Gamma\}$ is determined uniquely up to an isomorphism.

Proof. Let us start by defining the Hilbert space H_1, which we take to be an extension of the Banach space $V = L^1(\mathbb{R}, H_0) \cap L^2(\mathbb{R}, H_0)$ of measurable H_0 valued functions ϕ with

$$\|\phi\|_V := \left(\int_{-\infty}^{\infty} \|\phi(t)\|^2\, dt\right)^{1/2} + \int_{-\infty}^{\infty} \|\phi(t)\|\, dt < \infty. \tag{7.4}$$

Let $(\cdot,\cdot)_{H_0}$ denote the inner product on H_0 and define on V a quadratic form

$$\langle \phi, \psi\rangle := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\phi(t), a_e(t-\tau)\,\psi(\tau))_{H_0}\, dt d\tau \tag{7.5}$$

$$= 2\int_{-\infty}^{\infty} (\phi(t), \alpha_\infty \psi(t))_{H_0}\, dt + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\phi(t), \alpha_e(t-\tau)\,\psi(\tau))_{H_0}\, dt d\tau, \tag{7.6}$$

which is positive semi-definite by virtue of the power dissipation condition. However, there may be null vectors in V, that is vectors ϕ with $\langle \phi, \phi \rangle = 0$. Let N denote the set of null vectors and define $H_1 = \overline{V/N}$, where " $\bar{\cdot}$ " denotes closure in the norm inherited from the inner product. Then H_1 is a Hilbert space, whose inner product we also denote by $\langle \cdot, \cdot \rangle$.

Any $\phi \in V$ defines a unique element $[\phi] \in V/N \subset H_1$ with $\|[\phi]\|_{H_1}^2 = \langle \phi, \phi \rangle$. In particular, $[\phi]$ is zero if and only if $\langle \phi, \phi \rangle = 0$. Furthermore, the map $\phi \mapsto [\phi]$ is bounded from V into H_1 since

$$\|[\phi]\|_{H_1}^2 \leqslant 2 \|\alpha_\infty\| \int_{-\infty}^{\infty} \|\phi(t)\|^2 \, dt + \sup_\tau \|\alpha_e(\tau)\| \left(\int_{-\infty}^{\infty} \|\phi(t)\| \, dt \right)^2 \lesssim \|\phi\|_V^2 . \quad (7.7)$$

Thus a convergent sequence ϕ_j in V gives rise to a convergent sequence $[\phi_j]$ in H_1 and

$$\lim_j [\phi_j] = \left[\lim_j \phi_j \right]. \quad (7.8)$$

We define the operator Ω_1 to be the self adjoint generator of the one parameter unitary group of time translations. Specifically, we note that the transformations T_s of V into itself given by

$$T_s \phi(t) = \phi(t - s) \quad (7.9)$$

form a group which preserves the pre-inner product (7.5). Therefore, $s \mapsto T_s$ extends to a one parameter unitary group $s \mapsto U_s$ on H_1, which is in fact strongly continuous (as follows from strong continuity of T_s on V). The Stone–von Neumann theorem implies there is a unique self-adjoint operator Ω_1 on H_1 with $U_s = \mathrm{e}^{is\Omega_1}$.

Clearly there is a connection between Ω_1 and the operation of differentiation on V. To understand this, note that given $\phi \in V$ we have $[\phi] \in \mathcal{D}(\Omega_1)$ if and only if

$$\Omega_1 [\phi] = \lim_{s \to 0} \frac{U_s [\phi] - [\phi]}{is} = -i \lim_{s \to 0} \left[\frac{T_s \phi - \phi}{s} \right] \quad (7.10)$$

exists. Thus a sufficient condition for $[\phi]$ to be in $\mathcal{D}(\Omega_1)$ is for $s^{-1}(T_s \phi - \phi)$ to converge in V, which holds if and only if $\phi \in \mathcal{D}(\partial_t)$, in which case

$$\lim_s \frac{T_s \phi - \phi}{s} = -\partial_t \phi. \quad (7.11)$$

Therefore we have

$$\Omega_1 [\phi] = i [\partial_t \phi] \text{ for } \phi \in \mathcal{D}(\partial_t). \tag{7.12}$$

Note, however, that $[\phi] \in \mathcal{D}(\Omega_1)$ does not necessarily imply $\phi \in \mathcal{D}(\partial_t)$.

If we formally define $\Gamma^\dagger$ to be the map $\Gamma^\dagger v = [v\delta(\cdot)]$, ignoring for the moment that $v\delta(\cdot) \notin V$, we may calculate that

$$\Gamma[\phi] = \int_{-\infty}^{\infty} a_e(-t)\, \phi(t)\, dt, \tag{7.13}$$

and therefore

$$\Gamma e^{-it\Omega_1} \Gamma^\dagger v = \int_{-\infty}^{\infty} a_e(-s)\, v\delta(s+t)\, dt = a_e(t) v. \tag{7.14}$$

If $\alpha_\infty = 0$ then in fact (7.13) defines a bounded operator, and the above calculation may be justified. In that case, we could have started with a space V including point measures so that $v\delta(t)$ for $v \in H_0$ would be in V and we would have $\Gamma^\dagger v = [v\delta(\cdot)]$.

However, to consider also $\alpha_\infty \neq 0$, we work indirectly by defining the bounded map $S^\dagger : H_0 \to H_1$,

$$S^\dagger v := [vG]; \quad G(t) = i\mathrm{e}^t \begin{cases} 1, & t < 0, \\ 0, & t > 0. \end{cases} \tag{7.15}$$

Note that formally, $\left(\Omega_1 - i I_{H_1}\right) S^\dagger v = [v\delta(\cdot)]$, since $\{i\partial_t - i\}\, G(t) = \delta(t)$. To proceed rigorously, let us compute $S := \left(S^\dagger\right)^\dagger$. Given $\phi \in V$

$$(v, S[\phi]) = \left\langle S^\dagger v, [\phi] \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{G}(t)\, (v, a_e(t-\tau)\, \phi(\tau))\, dt\, d\tau, \tag{7.16}$$

so

$$\begin{aligned} S[\phi] &= \int_{-\infty}^{\infty} \left(-i \int_{-\infty}^{0} e^t a_e(t-\tau)\, dt \right) \phi(\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} \left(-i \int_{\tau}^{\infty} e^{\tau - t} a_e(-t)\, dt \right) \phi(\tau)\, d\tau. \end{aligned} \tag{7.17}$$

If we define $\Gamma : \mathcal{D}(\Omega_1) \to H_0$ by

$$\Gamma := S\left(\Omega_1 + i I_{H_1}\right), \tag{7.18}$$

then we recover (7.13) for $\phi \in \mathcal{D}(\partial_t)$,

$$\begin{aligned} \Gamma[\phi] &= \int_{-\infty}^{\infty} \left(\int_{\tau}^{\infty} e^{\tau - t} a_e(-t)\, dt \right) \{\partial_\tau + 1\} \phi(\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} a_e(-\tau)\phi(\tau)\, d\tau = 2\alpha_\infty \phi(0) + \int_{-\infty}^{\infty} \alpha_e(-\tau)\phi(\tau)\, d\tau, \end{aligned} \tag{7.19}$$

since $\phi \in \mathcal{D}(\partial_t)$ implies ϕ is continuous (so $\phi(0)$ is unambiguous).

One may easily verify that

$$\begin{aligned} S e^{-it\Omega_1} S^\dagger &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|t+s|} a_e(s)\, ds \\ &= e^{-|t|}\alpha_\infty + \frac{1}{2}\int_{-\infty}^{\infty} e^{-|t+s|}\alpha_e(s)\, ds, \end{aligned} \tag{7.20}$$

and therefore

$$\left(-\frac{d^2}{dt^2} + 1\right) S e^{-it\Omega_1} S^\dagger = 2\alpha_\infty \delta(t) + \alpha_e(t) = a_e(t). \tag{7.21}$$

Thus, with Φ_R defined by (7.1) and using (7.18) and (7.20) we get

$$\begin{aligned} \Gamma e^{-it\Omega_1}\left(\Gamma \Phi_R^2\right)^\dagger &= \left(-\frac{d^2}{dt^2} + 1\right) \Gamma \frac{1}{\Omega_1^2 + I_{H_1}} e^{-it\Omega_1}\left(\Gamma \Phi_R^2\right)^\dagger \\ &= \left(-\frac{d^2}{dt^2} + 1\right) S e^{-it\Omega_1} \Phi_R^2 S^\dagger \overset{\mathcal{D}}{\to} a_e(t) \qquad \text{as } R \to \infty, \end{aligned} \tag{7.22}$$

where $\overset{\mathcal{D}}{\to}$ denotes limit in the sense of distributions.

The uniqueness up to isomorphism can be understood as follows. Let $\{H_1, \Omega_1, \Gamma\}$ and $\left\{H_1', \Omega_1', \Gamma'\right\}$ be distinct representations, and suppose that

$$\mathcal{S} = \left\langle (\Gamma f(\Omega_1))^\dagger v:\ f \in C_c(\mathbb{R}),\, v \in H_0 \right\rangle \tag{7.23}$$

is dense in H_1. We denote by $\hat{f}$ the Fourier transform of $f \in C_c(\mathbb{R})$, so that

$$f(\Omega_1) = \int_{-\infty}^{\infty} \hat{f}(t)\, e^{it\Omega_1}\, dt. \tag{7.24}$$

Then, given $f, g \in C_c(\mathbb{R})$ and $v, w \in H_0$, we see that

$$\begin{aligned}
&\left\langle (\Gamma g(\Omega_1))^{\dagger} w, (\Gamma f(\Omega_1))^{\dagger} v \right\rangle_{H_1} \\
&= \lim_{R\to\infty} \left(w, \Gamma g(\Omega_1) f(\Omega_1)^{\dagger} \left(\Gamma \Phi_R^2\right)^{\dagger} v \right) \\
&= \lim_{R\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{g}(t)\, \hat{f}^*(\tau) \left(w, \Gamma e^{i(\tau - t)\Omega_1} \left(\Gamma \Phi_R^2\right)^{\dagger} v \right) dt d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{g}(t)\, \hat{f}^*(\tau)\, (w, a_e(t-\tau)\, v)\, dt d\tau \qquad (7.25) \\
&= \left\langle \left(\Gamma' g\left(\Omega_1'\right)\right)^{\dagger} w, \left(\Gamma' f\left(\Omega_1'\right)\right)^{\dagger} v \right\rangle_{H_1'}.
\end{aligned}$$

Thus, defining

$$T(\Gamma f(\Omega_1))^{\dagger} v = \left(\Gamma' f\left(\Omega_1'\right)\right)^{\dagger} v \tag{7.26}$$

and extending T to $\mathcal{S}$ by linearity, we produce a well defined isometry $T : \mathcal{S} \hookrightarrow H_1'$. The closure of this map, also denoted T, is an isometric imbedding $T : H_1 \hookrightarrow H_1'$. It is now easy to verify that

$$f(\Omega_1) = T^{\dagger} f\left(\Omega_1'\right) T, \qquad \text{for } f \in C_c(\mathbb{R}); \quad \Gamma' = \Gamma T^{\dagger}, \tag{7.27}$$

and thus the representation $\{H_1, \Omega_1, \Gamma\}$ is isomorphic to the restriction of $\left\{H_1', \Omega_1', \Gamma'\right\}$ to the closure of

$$\left\langle \left(\Gamma' f\left(\Omega_1'\right)\right)^{\dagger} v : \ f \in C_c(\mathbb{R}), v \in H_0 \right\rangle. \quad \blacksquare \tag{7.28}$$

7.2. Herglotz–Nevanlinna Theorems

In Section 2, we have already presented a more or less complete proof of Theorem 3.13 based on the classical Nevanlinna theorem 3.8 and the Naimark representation 3.12 of generalized spectral families. Since theorem 3.16 is proved in a very similar fashion, we present only a somewhat streamlined proof here.

Proof of Theorem 3.16. By the Herglotz–Nevanlinna theorem, for each $v \in H_0$ there is a finite measure $d\tilde{N}_{v,v}$ and a real number $\xi_{v,v}$ such that

$$(v, G(\zeta)\, v) = \xi_{v,v} + \int_{-\infty}^{\infty} \frac{1+\sigma\zeta}{\sigma-\zeta} d\tilde{N}_{v,v}(\sigma)\,. \tag{7.29}$$

The θ term drops out of the representation because $\zeta^{-1} G(\zeta) \to 0$ as $\zeta \to \infty$. In fact, $\xi_{v,v} = \mathrm{Re}\,(v, G(i)\, v)$. As in the derivation of Theorem 3.13, we define "off-diagonal" measures $d\tilde{N}_{v,w}$ for each pair $v, w \in H_0$ by polarization so that

$$(v, G(\zeta)\, w) = (v, \mathrm{Re}\, G(i)\, w) + \int_{-\infty}^{\infty} \frac{1+\sigma\zeta}{\sigma-\zeta} d\tilde{N}_{v,w}(\sigma)\,. \tag{7.30}$$

As above there is a "generalized spectral family" $K(\sigma)$ satisfying the hypothesis of the Naimark theorem 3.12 such that

$$(v, K(\sigma)\, w) = \int_{(-\infty,\sigma]} d\tilde{N}_{v,w}(\sigma)\,. \tag{7.31}$$

We denote by H_1 and $E(\sigma)$ the Hilbert space and resolution of the identity guaranteed by the Naimark theorem, letting T denote the associated mapping $T\colon H_1 \to H_0$. Thus, we have

$$G(\zeta) = \mathrm{Re}\, G(i) + \int_{-\infty}^{\infty} \frac{1+\sigma\zeta}{\sigma-\zeta} T\, dE(\sigma)\, T^{\dagger} \tag{7.32}$$

or

$$G(\zeta) = \mathrm{Re}\; G(i) + T \frac{I_{H_1} + \Omega_1\zeta}{\Omega_1 - \zeta I_{H_1}} T^{\dagger}, \quad \Omega_1 = \int_{-\infty}^{\infty} \sigma\, dE(\sigma)\,. \tag{7.33}$$

From this we may easily compute the following formula:

$$G(\zeta) = \operatorname{Re} G(iR) + T \frac{R^2 I_{H_1} + \Omega_1 \zeta}{\Omega_1 - \zeta I_{H_1}} \frac{I_{H_1} + \Omega_1^2}{R^2 I_{H_1} + \Omega_1^2} T^{\dagger}. \tag{7.34}$$

Therefore, given $v \in H_0$,

$$G(\zeta) v = \lim_{R \to \infty} \Gamma \frac{1}{\Omega_1 - \zeta I_{H_1}} \left(\Gamma \frac{R^2}{R^2 I_{H_1} + \Omega_1^2} \right)^{\dagger}, \quad \Gamma = T \sqrt{I_{H_1} + \Omega_1^2}, \tag{7.35}$$

because we have

$$\lim_{R \to \infty} \{\operatorname{Re}\, G(iR)\} v = 0 \tag{7.36}$$

by assumption, and

$$\lim_{R \to \infty} T \frac{\Omega_1 \zeta}{\Omega_1 - \zeta I_{H_1}} \frac{I_{H_1} + \Omega_1^2}{R^2 I_{H_1} + \Omega_1^2} T^{\dagger} v = 0 \tag{7.37}$$

for each ζ in the upper half plane, since $\frac{I_{H_1} + \Omega_1^2}{R^2 I_{H_1} + \Omega_1^2} \to 0$ strongly. ∎

APPENDIX A. STIELTJES INVERSION FORMULA AND NAIMARK'S THEOREM

In Section 3—and also in the examples of Section 4—we discussed the construction of conservative extensions. Reference was made there to two classical constructions—the Stieltjes Inversion formula and Naimark's construction for generalized spectral measures—which quite generally provide an explicit description of the Hilbert space H_1 in the operator versions of the Herglotz–Nevanlinna theorems. For completeness we include a discussion of those results here.

Stieltjes Inversion Formula

The Nevanlinna Theorem 3.8—in particular the relation (3.15)—suggests the introduction of the so-called *Cauchy transform* defined for complex-valued measures of finite variation on $\mathbb{R}$:

$$\tilde{N}(\zeta) = \int_{-\infty}^{\infty} \frac{dN(\sigma)}{\sigma - \zeta}, \quad \operatorname{Im}\, \zeta \neq 0. \tag{A.1}$$

The Nevanlinna theorem states that the set of functions which are Cauchy transforms of *non-negative* finite measures is exactly the class of analytic maps of the upper half plane into itself which decay as $\mathcal{O}(1/\text{Im }\zeta)$ as $\text{Im }\zeta \to \infty$. There does not seem to be such a simple description of the set of Cauchy transforms of complex measures. Nonetheless, a complex measure is uniquely determined by its Cauchy transform (ref. 10, Section 32.7, Lemma 4, ref. 7).

Proposition A.1. The Cauchy transform (A.1) is one-to-one, i.e. a complex measure of finite variation is uniquely determined by its Cauchy transformation. Furthermore, if $dN(\sigma)$ is a real (signed) measure of finite variation it can be recovered from its Cauchy transform $\tilde{N}(\zeta)$ restricted to $\{\text{Im }\zeta > 0\}$ by Stieltjes' formula:

$$\frac{N(\sigma_1+0)+N(\sigma_1-0)}{2} - \frac{N(\sigma_0+0)+N(\sigma_0-0)}{2} = \\ = \lim_{\eta\to+0} \frac{1}{\pi} \int_{\sigma_0}^{\sigma_1} \text{Im}\tilde{N}(\sigma+i\eta)\, d\sigma. \tag{A.2}$$

If $n(\sigma)\,d\sigma$ is the absolutely continuous component of the measure $dN(\sigma)$, in particular if $dN(\sigma) = n(\sigma)\,d\sigma$, we also have

$$n(\sigma) = \lim_{\eta\to+0} \frac{1}{\pi} \text{Im }\tilde{N}(\sigma+i\eta) \qquad \text{for Lebesgue almost every } \sigma. \tag{A.3}$$

Remark A.2. Another manifestation of (A.2), is the weak convergence

$$\operatorname*{wklim}_{\eta\to 0} \frac{1}{\pi} \text{Im}\tilde{N}(\sigma+i\eta)\, d\sigma = dN(\sigma), \tag{A.4}$$

that is,

$$\int_{-\infty}^{\infty} f(\sigma)\, dN(\sigma) \\ = \lim_{\eta\to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} f(\sigma)\, \text{Im }\tilde{N}(\sigma+i\eta)\, d\sigma \ \text{ for all } f \in C_c(\mathbb{R}). \tag{A.5}$$

Thus a scalar Herglotz function $g(\zeta)$ which is $\mathcal{O}(1/\mathrm{Im}\ \zeta)$ as $\mathrm{Im}\ \zeta \to \infty$ may be represented by the formula

$$g(\zeta) = \lim_{\eta \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\sigma - \zeta} \mathrm{Im}\ g(\sigma + i\eta)\, d\sigma. \tag{A.6}$$

In particular, if $\mathrm{Im}\ g(\sigma + i0) = \lim_{\eta \to 0} \mathrm{Im}\ g(\sigma + i\eta)$ exists for almost every σ and

$$\lim_{\eta \to 0} \int_{-\infty}^{\infty} |\mathrm{Im}\ g(\sigma + i\eta) - \mathrm{Im}\ g(\sigma + i0)|\, d\sigma = 0, \tag{A.7}$$

then

$$g(\zeta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\sigma - \zeta} \mathrm{Im}\ g(\sigma + i0)\, d\sigma. \tag{A.8}$$

In general, however there may be a singular component to the measure dN in the representation (3.15).

For an operator valued Herglotz function $G(\zeta)$, with $\|G(\zeta)\| = \mathcal{O}(1/\mathrm{Im}\ \zeta)$, we have therefore

$$G(\zeta) = \lim_{\eta \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\sigma - \zeta} \mathrm{Im}\ G(\sigma + i\eta)\, d\sigma \tag{A.9}$$

with the integral understood in the weak sense, i.e.

$$(v, G(\zeta)\, w) = \lim_{\eta \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\sigma - \zeta} (v, \mathrm{Im}\ G(\sigma + i\eta)\, w)\, d\sigma, \ \text{ for } v, w \in H_0. \tag{A.10}$$

Thus the generalized spectral family $K(\sigma)$ associated to $G(\zeta)$ can be expressed through the formula

$$d(v, K(\sigma)\, w) = \operatorname*{wklim}_{\eta \to 0} \frac{1}{\pi} (v, \mathrm{Im}\ G(\sigma + i\eta)\, w)\, d\sigma. \tag{A.11}$$

A.2. Naimark's Theorem

The Naimark construction for $K(\sigma)$, which leads to Theorem 3.12, is most easily understood by realizing H_1 as the Hilbert space $L^2(dK)$, where the latter space needs to be appropriately defined. Formally $L^2(dK)$ should consist of all H_0 valued functions Ψ such that $\int_{-\infty}^{\infty}(\Psi(\sigma), dK(\sigma)\Psi(\sigma))$ is finite, modulo null functions (for which the integral is 0). It is not always clear how to make sense of this integral however, and one must turn to a more abstract definition of $L^2(K)$. However, if $dK(\sigma)=m(\sigma)\,d\sigma$, with $m(\sigma)$ bounded for almost every σ, we can avoid the abstract construction by defining $H_1=L^2(dK)$ to be the space of H_0 valued functions Ψ such that

$$\|\Psi\|_K^2=\int_{-\infty}^{\infty}(\Psi(\sigma), m(\sigma)\Psi(\sigma))\,d\sigma<\infty, \tag{A.12}$$

modulo null functions for which $\|\Psi\|_K^2=0$. Note that $m(\sigma)$ is necessarily a positive operator since

$$(v, m(\sigma)v)=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\sigma-\varepsilon}^{\sigma+\varepsilon} d(v, K(\sigma)v). \tag{A.13}$$

For more general measures dK there are two options. We could define the norm appearing in (A.12) for H_0 valued *simple functions*—functions taking a finite number of values—and let H_1 be the closure of the space of simple functions under this norm. Alternatively, we can express $dK(\sigma)$ as the weak limit (A.11) and define $H_1=L^2(dK)$ as the space of functions Ψ such that

$$\|\Psi\|_K^2=\lim_{\eta\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty}(\Psi(\sigma), \operatorname{Im} G(\sigma+i\eta)\Psi(\sigma))\,d\sigma \tag{A.14}$$

exists and is finite, modulo null functions (as always)

However it may be defined, once the space $H_1=L^2(dK)$ has been constructed, there is a natural spectral measure $E(\sigma)$ given by

$$[E(\sigma)\Psi](\nu)=\begin{cases}\Psi(\nu) & \text{if } \nu\leqslant\sigma,\\ 0 & \text{if } \nu>\sigma,\end{cases} \tag{A.15}$$

that is $E(\sigma)$ corresponds to multiplication by the characteristic function of $(-\infty,\sigma]$. The associated self adjoint operator $\Omega_1=\int_{-\infty}^{\infty}\sigma\,dE(\sigma)$ is simply multiplication by the independent variable:

$$\Omega_1\Psi(\sigma)=\sigma\Psi(\sigma). \tag{A.16}$$

Finally there is a natural map $\Gamma^\dagger\colon H_0 \to H_1$ which takes an element $\psi \in H_0$ to the constant function with value ψ:

$$\left[\Gamma^\dagger \psi\right](\sigma) = \psi \qquad \text{for every } \sigma \in \mathbb{R}. \tag{A.17}$$

It is easy to verify that

$$K(\sigma) = \Gamma E(\sigma)\, \Gamma^\dagger. \tag{A.18}$$

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